Competing Auction Houses*

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Abstract

We consider a model where sellers make repeated attempts to sell an object via two competing auction houses. An auction house that attracts a seller runs a Vickrey auction among a random sample of buyers and collects two fees: a listing fee and, if the object is sold, a closing fee. We characterize equilibria and show that two non-identical auction houses may coexist in equilibrium if the population of sellers is sufficiently differentiated in their time preferences. In such an equilibrium impatient, "amateur" sellers choose the more popular (the one that attracts more bidders) and more expensive auction house, while patient, "professional" sellers choose the less popular and cheaper one.

Keywords: Competing auctions, mediator, listing fee, closing fee

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1 Introduction

This paper considers a model of Internet-style trade, where a seller cannot deal directly with buyers, instead, the trade must be mediated. A mediator sets the rules of a trade procedure (an auction mechanism) and collects fees from the traders. This is the case in Internet auctions where the role of mediators is played by such giant commercial institutions as *eBay* and, previously, *Amazon* and *Yahoo*.

In the last ten years, *eBay* took over more than fifteen Internet auction houses that operated on local as well as international markets.[1] Moreover, its main competitors, *Yahoo* and *Amazon*, discontinued their Internet auction service (*Yahoo* on June 16, 2007 and *Amazon* on September 8, 2008), leaving *Naspers* as, probably, the only serious player other than *eBay* on this market.[2]

The main motivation for this paper is the following question: *Will one auction house (e.g., eBay) eventually monopolize the Internet auction market, or can competitors coexist on this market?*

In our framework, two competing auction houses try to attract a single seller drawn at random from a heterogeneous pool of sellers. The seller has one object for sale and every bidder demands one object. We assume that auction houses sell objects via Vickrey auction with reserve price and receive revenue by charging sellers two fees: a *listing fee*, a fixed amount paid by a seller regardless of the auction outcome, and a *closing fee*, a percentage of the closing price if the object is sold. Competing auction houses differ

[1]The list of eBay's acquisitions includes Up4Sale.com (US), Butterfield & Butterfield (US), Alando (Germany), Half.com (US), Internet Auction Co. (South Korea), iBazar (France), CARad.com, EachNet (China), Baazee.com (India), Marktplaats.nl (Netherlands), Shopping.com (US), Tradera (Sweden), StubHub (US), GittiGidiyor (Turkey), StumbleUpon (Canada), Afterbuy (Germany), GMarket (South Korea).

[2]Naspers operates in 13 countries where eBay is not present or not a market leader and includes such brands as Aucro (Bulgaria, Czech Republic, Romania, Slovakia, Ukraine), Allegro (Poland), QXL (Denmark, Norway, and, formerly, UK), Ricardo (Switzerland), Molotok (Russia).

in their number of bidders and service fees. Since sellers are heterogeneous (characterized by a private discount factor and a private object value), a seller's choice of an auction house depends on his type. There are three important features in our model. First, auction fees are fixed and publicly known: once announced, they cannot be altered later on. Second, whenever a seller fails to sell his object, he is allowed to offer it for (re-)auction again, as many times as he wants. Third, in every auction the seller faces a different set of bidders drawn from a large population. In addition, all players are assumed to be risk neutral, and auction houses are at least as patient as the sellers (in their preferences over time).

Our analysis provides a new insight regarding optimal selection of auction fees. The first result shows that a positive listing fee is never a best reply for an auction house, and, consequently, *in any equilibrium the listing fee must be equal to zero*. The intuition behind this result is that when an auction house is more patient than a seller, there is a distortion between their interests that is minimized when the listing fee is zero. Thus, we show that an auction house's policy is effectively one-dimensional: it is the choice of its closing fee only.

Second, we characterize Bayesian Nash equilibria in our model and show that there exists at most one equilibrium which, depending on the model parameters, may be one of the following types: *(contestable) monopoly* or *market segmentation*. A monopoly equilibrium arises if the pool of sellers is not too differentiated. It is a result of a standard Bertrand competition, where a more *popular* auction house (that attracts more bidders) sets fees low enough that sellers of every type will be attracted to that auction house, and the competitor is forced to "leave" the market.

A contestable monopoly is probably the most expected situation on the market. However, if sellers are sufficiently differentiated, a *market segmentation* equilibrium obtains, where auction houses "split the market." This

situation takes place if the more popular auction house can obtain higher expected payoffs by attracting only sellers of one type (rather than all types, as it is in the contestable monopoly). This allows its competitor to set low positive fees, attract sellers of the other type, and obtain a positive expected payoff as well. This equilibrium is a result of "product differentiation" where sellers are discriminated on the basis of their time preferences. Less patient sellers are naturally attracted to the more popular auction house that provides a larger number of bidders immediately. In contrast, more patient sellers are not so constrained by time, so that they can afford to re-auction their objects a few times on the less popular but cheaper auction house.

Ellison, Fudenberg, and Möbius (2004) in a closely related article (see also Moldovanu, Sela, and Shi, 2008) find that whenever two auction houses co-exist in equilibrium, the "law of one price" (i.e., the same expected closing price and the same buyer/seller ratio across competing auctions) should hold. This result relies on the assumption that bidders can freely choose between auction houses. Contrarily to Ellison, Fudenberg, and Möbius (2004), Brown and Morgan (2009) demonstrate in field experiments that eBay prices were consistently higher than those on Yahoo, and eBay attracted more buyers per seller. In attempt to explain this observation, we depart from Ellison, Fudenberg, and Möbius' (2004) setting by assuming that bidders are immobile, i.e., the number of bidders that an auction house attracts is given exogenously. There are other reasons why such an assumption may be more plausible. On the one hand, customers of one auction house may be unaware of existence of a competitor. On the other hand, in reality an auction house takes certain actions to make participants accustomed to its features and design in order to make transition to a competitor costly, for instance, using bonuses for participation or tracking reputation. Our *market segmentation* suggests an explanation for Brown and Morgan's (2009) findings.

There are two other important assumptions in our model: infinite re-

auction attempts for a seller and a new set of bidders in every auction. A seller has a re-auction option in real life and this option has essential impact on players' strategic behavior, as noted, for example, by Fudenberg at al. (1985), Milgrom (1987), McAfee and Vincent (1997), Horstmann and LaCasse (1997), Gupta and Lebrun (1999). The second assumption – that the seller faces a different set of bidders drawn from a large population in each period – is reasonable in the context of Internet auctions where a typical auction runs several days, and most of the bids are received in the very last day. Our model can be considered as an instance for many similar sales on Internet where a buyer's objective is to purchase an object of a certain kind, not to purchase an object from a specific seller. A buyer who fails to buy an object from a seller can obtain it elsewhere and therefore has no reason to return to this particular seller. Hence, the seller who re-auctions his object believes that bidders who participated in his previous auctions would not come again. In contrast, a large part of the existing literature on auctions with resale assumes that there is the same set of bidders in all auctions.[3] This implies two differences from our model. In models with a possibility of one-time after-auction resale, each bidder places a positive probability on buying in a secondary market if she loses the auction (Gupta and Lebrun, 1999; Haile 1999, 2000, 2001, 2003; Zheng 2002; Krishna, 2002, Section 4.4; Calzolari and Pavan 2006; Garatt and Tröger 2006; Pagnozzi 2007). In models with re-auctioning, the optimal reserve price declines due to Bayesian updating of the distribution of bidders' private values after every auction (Fudenberg at al. 1985; McAfee and Vincent 1997). The latter effect does not appear in our model, because we assume that the auction houses

[3]The exceptions are Haile (1999, 2001) who allows new bidders (in particular, *all* new bidders) to participate in a re-auction, and Matros and Zapechelnyuk (2008) who consider a more restrictive setting with a monopoly auction house and a unidimensional type of sellers. Different, but related settings are considered by Bikhchandani and Huang (1989), Bose and Deltas (1999, 2007) and Calzolari and Pavan (2006) who model resale to a given secondary market where the original bidders need not participate.

cannot change their fee policy during the game. As a result, the optimal seller's strategy (the choice of the auction house and the reserve price) is stationary.

In our model a winning bidder is not allowed to re-auction the object. This is a simplifying assumption which can be relaxed without any effect on the results: since a new set of bidders arrives in each period, there is no issue of signaling and information communication for the bidders between periods (in contrast to Haile 1999, 2000, 2001, 2003; Zheng 2002; and others). If a winning bidder becomes a seller, she would face ex-ante the same environment in the next-period auction. The expected revenue from a new auction is not higher than her current use value, thus she prefers to consume the object. This contrasts our results, in particular, to Zheng (2002) who assumes that a fixed, finite set of bidders is involved in trade, where, despite that bidders are ex-ante symmetric, the initial seller and the winning bidder face different trade environments, and the winner may benefit from a re-auction.

The paper is organized as follows. The model is described in Section 2. In Sections 3 and 4 we depict the optimal behavior of the seller and auction houses. Section 5 characterizes all equilibria. Section 6 concludes. The Appendix contains omitted proofs.

2 The Model

Let $\mathcal{N}$ be a large (infinite) population of bidders and $\mathcal{M}$ be a large (infinite) population of sellers. Every bidder $i \in \mathcal{N}$ is characterized by her private *use value* $v_i \in [\underline{v}, \overline{v}]$ of the object. All private use values are independent and identically distributed according to distribution function F. Every seller $s \in \mathcal{M}$ is characterized by two independent private parameters: his use value $v_s \in [\underline{v}, \overline{v}]$ (sellers' use values are independent and identically distributed according to distribution function F_s) and discount factor δ_s. We assume that δ_s has

either low (δ_L) or high (δ_H) value with probability α and $(1-\alpha)$ respectively, where $0 < \delta_L \leq \delta_H < 1$ and $\alpha \in [0,1]$. We also assume that functions F and F_s are differentiable and have positive density on $(\underline{v}, \overline{v})$, and, in addition, satisfy some technical conditions. Namely, F satisfies the monotonic hazard rate condition (e.g., Myerson 1981) and F_s satisfies a similar condition, that is, $z - \frac{1-F(z)}{f(z)}$ and $F_s(z) - (\overline{v} - z)f_s(z)$ are strictly increasing on $(\underline{v}, \overline{v})$, where f and f_s denote the corresponding density functions. Further, there are two auction houses $j = 1, 2$, each of which is characterized by the number of bidders n_j that it attracts during an auction. Distribution functions F and F_s and parameters α, δ_L, δ_H, n_1 and n_2 are common knowledge. We also assume that all players are risk neutral.

The timing of the game is as follows. In period 0, the two auction houses simultaneously announce fees for all subsequent auctions: listing fees, $c_j \geq 0$, and closing fees, a fraction $\mu_j \in [0,1]$ of the closing price. The fees are commitments that cannot be altered in later periods. Then a seller s is drawn at random from population $\mathcal{M}$.

In every period $t = 1, 2, \ldots$ (as long as the object is not sold or consumed), the seller either consumes the object (and the game ends) or chooses an auction house $j \in \{1, 2\}$ to place his object for sale. In the latter case, the seller announces a reserve price, and a Vickrey auction is run among a random sample of n_j bidders from population $\mathcal{N}$. As a result of the auction, the object is transferred to a winner (the highest bidder), who pays to the seller the price equal to the second highest bid (or the reserve price), or the object is returned to the seller, if no bid is above the reserve price. Regardless of the auction outcome, the seller pays to auction house j the listing fee, c_j, and, in addition, if the object is sold, the closing fee, fraction μ_j of the closing price. If the object is sold, the game ends, otherwise it proceeds to the next period.

3 Seller's Decision Problem

In this section we describe and solve seller's decision problem. Consider a seller with use value $v_s \in [0,1]$ and discount factor $\delta_s \in \{\delta_L, \delta_H\}$. Given the stationarity of the environment, we will only examine seller's time-invariant (Markov) strategies.

The seller faces the fees of both auction houses, (c_1, μ_1) and (c_2, μ_2), and makes the following decisions. First, he makes an allocation decision (λ_1, λ_2), where λ_j is the probability that the object is auctioned off on auction house $j = 1, 2$, and $1 - \lambda_1 - \lambda_2$ is the probability that the object is consumed, $\lambda_1, \lambda_2 \in [0,1]$, $\lambda_1 + \lambda_2 \leq 1$. Second, for each auction house j, the seller chooses a reserve price r_j that he would use had he chosen auction house j to auction off his object.[4]

Since we assume that the object is being sold via Vickrey auction, every bidder's dominant strategy is to bid her true use value (e.g., Krishna 2002), thus the only parameter that affects the expected revenue from the auction is the reserve price. Let us analyze how the seller chooses the revenue-maximizing reserve price.

Suppose that the seller chooses auction house j for auctioning the object. For every reserve price r_j denote by $P_j(r_j)$ the *probability that the object is sold* on auction house j, and by $x_j(r_j)$ the *expected closing price*, that is, the expected payment of the winning bidder conditional on the event that the object is sold. Denote by $u_j(r_j, u^*)$ the expected seller's revenue from the auction j, where u^* is the highest continuation payoff that the seller expects to obtain in the next period if the object is not sold.[5] Thus,

$$u_j(r_j, u^*) = -c_j + (1 - \mu_j)P_j(r_j)x_j(r_j) + (1 - P(r_j))\delta_s u^*. \tag{1}$$

[4] A description of the seller's strategy must include the decision about the reserve price in auction house j whether or not the object is auctioned at that auction house.

[5] We assume that the seller does not derive any utility from the object before it is sold or consumed.

Then, an optimal reserve price $r_j(u^*)$ is a solution of the optimization problem

$$r_j(u^*) \in \arg\max_{r_j} u_j(r_j, u^*). \tag{2}$$

Lemma 1 *For every pair of auction fees (c_j, μ_j) and every $u^* \in [\underline{v}, \bar{v}]$, there exists a unique solution $r_j(u^*) \in [\underline{v}, \bar{v}]$ of the decision problem (2). It is a unique solution of the following equation*

$$r_j - \frac{1 - F(r_j)}{f(r_j)} = \frac{\delta_s u^*}{1 - \mu_j}, \text{ if } \delta_s u^* < (1 - \mu_j)\left(\bar{v} - \frac{1 - F(\bar{v})}{f(\bar{v})}\right), \tag{3}$$

and

$$r_j = \bar{v}, \text{ if } \delta_s u^* \geq (1 - \mu_j)\left(\bar{v} - \frac{1 - F(\bar{v})}{f(\bar{v})}\right).$$

The proof is in the Appendix.

Note that equation (3) is a standard first-order condition in a Vickrey auction with a reserve price (see, e.g., Krishna, 2002). The uniqueness of the solution follows from the hazard rate condition stipulating that $z - \frac{1-F(z)}{f(z)}$ is a strictly increasing function.

Let us now consider the seller's allocation decision (λ_1, λ_2). The seller makes an allocation decision (to consume the object or to auction it off on auction house 1 or 2) which maximizes his expected payoff. Note that since in every period the seller faces ex-ante the same environment, if he chooses auction house j once, then he will choose it in all periods until the object is sold.

Let u_j^* be the expected seller's payoff if he sells the object at auction house j in all periods. From (1) it follows that u_j^* is a solution of the following equation,

$$\begin{aligned} u_j^* &= u_j(r_j(u_j^*), u_j^*) \\ &\equiv \left(-c_j + (1 - \mu_j)P_j(r)x_j(r) + (1 - P(r))\delta_s u_j^*\right)\Big|_{r = r_j(u_j^*)} \end{aligned} \tag{4}$$

The next lemma states that equation (4) has a unique solution.

Lemma 2 *The mapping $u_j(r_j(\cdot), \cdot)$ has a unique fixed point.*

The proof is in the Appendix.

The seller chooses $(\lambda_1^*, \lambda_2^*)$ to maximize his expected revenue

$$(\lambda_1^*, \lambda_2^*) \in \arg\max_{\lambda_1, \lambda_2} \lambda_1 u_1^* + \lambda_2 u_2^* + (1 - \lambda_1 - \lambda_2) v_s. \tag{5}$$

We say that a seller *prefers* auction house i to auction house j if $u_i^* \geq u_j^*$. Thus, the seller auctions the object off at a preferred auction house and receives $\max\{u_1^*, u_2^*\}$ whenever this revenue exceeds his use value v_s, otherwise, he consumes the object and receives v_s.

Note that the expected payoff from auction j, u_j^*, and the optimal reserve price $r_j(u_j^*)$ do not depend on the seller's use value, v_s, but they depend on the seller's discount factor δ_s. Indeed, a more patient seller chooses a higher reserve price and receives a higher expected revenue from the auction. Therefore, sellers with the same discount factor have the same preference over auction houses, and only sellers with different discount factors may potentially prefer different auction houses. With a slight abuse of terminology, we will refer to a more (less) patient seller with discount factor, δ_H (δ_L), as an *H-type* (respectively, *L-type*) seller.

4 Auction Houses

Seller's behavior as a function of auction houses' fees was described in the previous section. In this section, we consider how auction houses choose their fees.

4.1 Payoffs

In auction house $j = 1, 2$, in each period a set of n_j bidders is randomly drawn to participate in the auction. The number of bidders n_j is fixed and

commonly known.[6] We assume that both auction houses have the same discount factor $\gamma \in (0,1]$ and are more patient than sellers, $\gamma \geq \delta_H (\geq \delta_L)$.[7] This assumption is economically justified: it is standard in the literature to assume that a business (an internet auction house) is more patient than an individual (a seller).

Fix auction houses' fees $a_1 = (c_1, \mu_1)$ and $a_2 = (c_2, \mu_2)$, the seller's use value v_s and his discount factor δ_θ, $\theta \in \{L, H\}$. Denote by $\lambda_j^\theta(a_1, a_2)$ the probability that a θ-type seller chooses auction house j, by $u_j^\theta(a_j)$ the seller's expected payoff from auctioning the object off on auction house j, and by $r_j^\theta(a_j)$ the optimal reserve price, $j = 1, 2$. The expected payoff of auction house j conditional on that a θ-type seller has chosen it to sell the object is

$$w_j^\theta(a_j) = c_j + \mu_j P_j(r_j^\theta(a_j)) x_j(r_j^\theta(a_j)) + (1 - P(r_j^\theta(a_j)))\gamma w_j^\theta(a_j). \tag{6}$$

The next lemma shows that the above equation has a unique solution.

Lemma 3 *For every $a_j = (c_j, \mu_j) \in \mathbb{R}_+ \times [0,1]$ and every $\theta \in \{L, H\}$, there exists a unique solution, $w_j^\theta(a_j)$, of equation* (6).

The proof is similar to the proof of Lemma 2 and thus omitted.

The unconditional payoff $\bar{w}_j(a_1, a_2)$ of auction house j is given by the product of the conditional payoff described above and the probability that a seller of each type chooses auction house j for selling the object

$$\bar{w}_j(a_1, a_2) = \alpha \lambda_j^L(a_1, a_2) w_j^L(a_j) + (1 - \alpha)\lambda_j^H(a_1, a_2) w_j^H(a_j). \tag{7}$$

Equation (7) shows that each auction house faces the following trade-off: lower fees lead, on the one hand, to a higher probability of attracting a seller

[6] The results can be generalized to the case where the number of bidders n_j is random, drawn from the same distribution in each period. Indeed, all that matters here is that a seller makes the decision of auctioning his object *before* n_j is drawn, thus his decision depends the distribution of the number of bidders (which is constant across periods), but not on its realizations.

[7] The assumption that auction houses share the same discount factor is not critical and can be relaxed without affecting the results.

(or even to stealing a seller from the competitor), but on the other hand, to a lower revenue from the attracted seller.

4.2 Closing-Fee Auctions

This section addresses and partially answers one of the main questions of this paper: *What are the optimal auction house fees?* We will show that if an auction house makes a positive profit in an equilibrium, the listing fee in that equilibrium must be zero.

Theorem 1 *Let (c_1, μ_1) and (c_2, μ_2) be arbitrary fees of the auction houses. If auction house i receives a positive expected payoff and $c_i > 0$, then (c_i, μ_i) is not a best reply of auction house i to the fees (c_j, μ_j), $j \neq i$.*

The proof is in the Appendix.

The intuition behind this result is as follows. Suppose that a seller has chosen auction house i. Since the seller is less patient than the auction house, he will always choose his reserve price lower than the reserve price which maximizes the expected payoff of the auction house.[8] Since $c_i > 0$, auction house i can simultaneously increase closing fee μ_i and decrease listing fee c_i such that the seller's expected revenue does not change, but his "virtual continuation value" $\frac{\delta_s v_s}{1-\mu_i}$ increases. By equation (3), a higher "virtual continuation value" leads to a higher reserve price, which in turn increases the expected revenue of the auction house.

It immediately follows from Theorem 1 that if an auction house has a positive expected payoff in an equilibrium, then its listing fee must be equal to zero. However, in an equilibrium where auction house i has zero payoff

[8]The assumption that the auction house is at least as patient as sellers is important. The result in Theorem 1 need not hold if a sufficient fraction of the sellers' population is more patient than the auction house.

(and thus no deviation can lead to a positive payoff), every pair of fees (c_i, μ_i) is a best reply, that is, all strategies of auction house i lead to the same zero payoff. To simplify notations and characterization of equilibria, we will assume that if auction house i has zero payoff in an equilibrium, then $c_i = 0$. This assumption and Theorem 1 give the following corollary.

Corollary 1 *In every equilibrium the listing fees of both auction houses are equal to zero.*

Corollary 1 allows us to fix listing fees at zero and describe the auction houses' strategies as the choice of *closing fees only.* In what follows, we will assume that the listing fees are zero. Thus, the notation for a strategy $a_j = (c_j, \mu_j)$ of auction house j will be replaced by μ_j and understood as $a_j = (0, \mu_j)$. Therefore, from (7), the unconditional payoff $\bar{w}_j(a_1, a_2) = \bar{w}_j(\mu_1, \mu_2)$ of auction house j can be rewritten as

$$\bar{w}_j(\mu_1, \mu_2) = \alpha\lambda_j^L(\mu_1, \mu_2)w_j^L(\mu_j) + (1-\alpha)\lambda_j^H(\mu_1, \mu_2)w_j^H(\mu_j). \tag{8}$$

4.3 Different Auction Houses

We say that auction house i is more *popular* than auction house j if it attracts more bidders, i.e., $n_i > n_j$. Without loss of generality, we assume that $n_1 \geq n_2$.

Since the total expected revenue from an auction depends on the number of bidders, it is easy to see that selling an object on the more popular auction yields higher expected closing price. Thus, if two auction houses charge equal fees, any seller (whether H-type or L-type) will prefer the more popular auction house, and hence the less popular one attracts no sellers and receives zero payoff.

Suppose that the auction houses are equally popular, $n_1 = n_2$. Since in an equilibrium auction houses compete only in closing fees, they are engaged

in the classic Bertrand competition: an auction house with a lower closing fee is more attractive to a seller of *any* type. It immediately follows that closing fees must be equal to zero in an equilibrium.

Proposition 1
(i) If the auction houses are equally popular, $n_1 = n_2$, then there exists a unique equilibrium where all fees are equal to zero, $(c_1, \mu_1) = (0, 0) = (c_2, \mu_2)$.
(ii) If auction house 1 is more popular, $n_1 > n_2$, then it sets a positive closing fee in equilibrium, $\mu_1 > 0$.

The proof is in the Appendix.

Part (i) of Proposition 1 completely characterizes the equilibria in the case of equally popular auction houses. In what follows, we will analyze the more complicated case, $n_1 > n_2$. In order to understand how the sellers of different types choose between auction houses that are not equally popular, consider the following diagram (Fig. 1).

Figure 1 illustrates the indifference curves of L and H types of sellers for the case of $n_1 > n_2$. For every closing fee μ_2 of auction house 2, let $\phi_\theta(\mu_2)$ denote the critical level of μ_1 such that with these fees a θ-type seller, $\theta = L, H$, is indifferent between the two auction houses. The graph $\{(\phi_\theta(\mu_2), \mu_2) : \mu_2 \in [0, 1]\}$ represents the indifference curve of a θ-type seller. Note that at the point $(\mu_1, \mu_2) = (1, 1)$ the indifference curves for both seller types coincide, since in this case the auction houses claim the entire surplus, leaving sellers with zero expected revenue, regardless of their types. Two curves $\phi_L(\mu_2)$ and $\phi_H(\mu_2)$ divide the closing fee space, $\mu_1 \times \mu_2$, into three areas:

A is the area where μ_1 is too high relative to μ_2, so that all sellers prefer auction house 2;

B is the area where sellers with different patience levels prefer different auction houses;

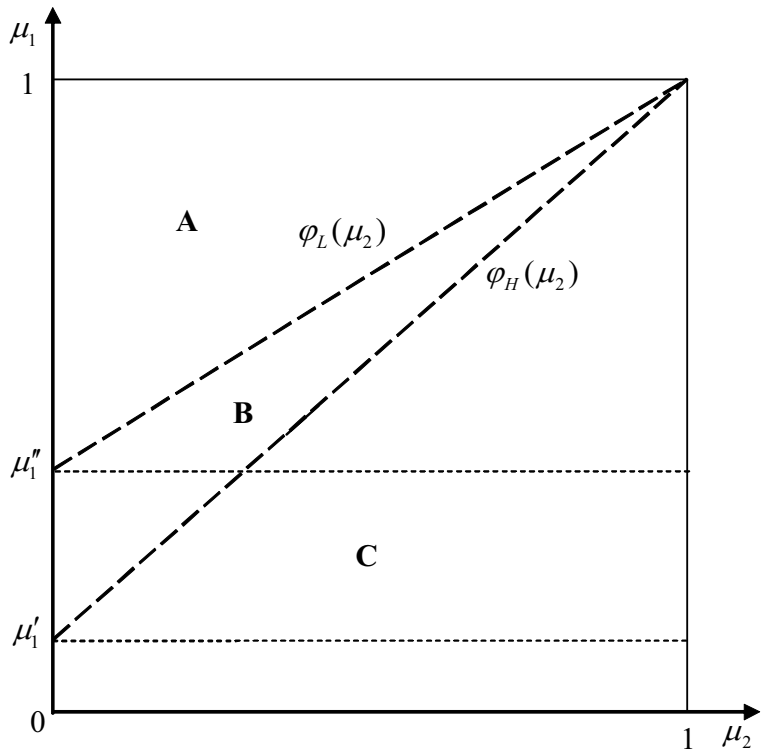

Figure 1: Indifference curves of L and H types of sellers

C is the area where μ_2 is too high relative to μ_1, so that all sellers prefer auction house 1.

It is important to note that the more popular auction house 1 can always guarantee to attract L-type sellers by setting its closing fee $\mu_1 \leq \mu_1''$. Furthermore, it can always guarantee to attract *all* sellers by setting its closing fee $\mu_1 \leq \mu_1'$, no matter what closing fee μ_2 is chosen by auction house 2. In contrast, the less popular auction house 2 cannot guarantee to attract any type of sellers.

It is clear from Figure 1 that if an H-type seller is indifferent between two auction houses, then L-type seller prefers the more popular auction house 1. The next proposition shows that this statement holds in general.

Proposition 2 *For every pair of fees, $(\mu_1, \mu_2) \neq (1, 1)$, if an H-type seller is indifferent between two auction houses, then L-type seller prefers the more popular auction house 1. Formally, if $n_1 > n_2$, then $\phi_H(\mu_2) < \phi_L(\mu_2)$ for every $\mu_2 < 1$.*

The proof is in the Appendix.

Intuitively, for an L-type seller (the impatient one), the possibility to obtain a higher revenue *right now* is the most important, and thus he receives a higher payoff from the more popular auction house. To see this, imagine the extremely impatient seller, $\delta_L = 0$, who obtains utility only from the current-period sale. In this case, only the current number of bidders on the auction matters for him. In contrast, a more patient seller can afford to wait and try to sell the object more than once, thus, eventually facing more bidders over time, even if there are a few of them in the current period.

5 Equilibrium Analysis

In this section we will show that if auction houses are not equally popular, $n_1 > n_2$, then there are three types of equilibria: *monopoly*, *contestable monopoly*, and *market segmentation*.

5.1 Monopoly

Suppose that auction house 1 is much more popular than auction house 2, so that auction house 1 is a monopoly on the market. For illustration, imagine the extreme case, $n_2 = 0$. In this extreme case a seller with use value v_s and discount factor δ_θ , $\theta = H, L$, will never choose auction house 2, i.e., $\lambda_2^\theta(\mu_1, \mu_2) = 0$ for every (μ_1, μ_2). Furthermore, the seller chooses auction

house 1 (as opposed to consuming the object) if $u_1^\theta(\mu_1) > v_s$.[9] Hence, for a given closing fee μ_1, the probability that a seller of type θ auctions the object off is equal to the probability that $v_s < u_1^\theta(\mu_1)$, that is, for every $\theta = H, L$ and every μ_1

$$\lambda_1^\theta(\mu_1, \mu_2) = F_s(u_1^\theta(\mu_1)).$$

From (8), the expected payoff of the monopolist is

$$\bar{w}_1(\mu_1, \mu_2) = \alpha\lambda_1^L(\mu_1, \mu_2)w_1^L(\mu_1) + (1-\alpha)\lambda_1^H(\mu_1, \mu_2)w_1^H(\mu_1)$$

$$= \alpha w_1^L(\mu_1)F_s(u_1^L(\mu_1)) + (1-\alpha)w_1^H(\mu_1)F_s(u_1^H(\mu_1)).$$

Therefore, auction house 1 solves the following maximization problem

$$w^M = \max_{\mu_1} \left[\alpha w_1^L(\mu_1)F_s(u_1^L(\mu_1)) + (1-\alpha)w_1^H(\mu_1)F_s(u_1^H(\mu_1))\right]. \tag{9}$$

The following lemma helps to establish uniqueness of the solution.

Lemma 4 *For every $i = 1, 2$ and every $\theta = H, L$, the expression*

$$w_i^\theta(\mu_i) \cdot F_s(u_i^\theta(\mu_i))$$

is strictly concave in μ_i.

The proof is in the Appendix. A solution of (9) is unique, since the sum of concave functions is concave. We will refer to this solution, denoted by $\mu_1^M \in [0, 1]$, as the *monopoly closing fee*. Further, the equilibrium where auction house 1 sets the monopoly closing fee and attracts all sellers will be called the *monopoly equilibrium.*

Proposition 3 *Let $n_1 > n_2$. The monopoly equilibrium exists if and only if*

$$\mu_1^M \leq \phi_H(0). \tag{10}$$

[9] A tie, $v_s = u_1^\theta(\mu_1)$, is a zero-probability event and thus can be ignored in the analysis.

Proof. Condition (10) means that H-type sellers prefer auction house 1 when $\mu_2 = 0$. From Proposition 2, $\phi_H(0) < \phi_L(0)$. Thus, condition (10) implies that both types of sellers prefer auction house 1 when $\mu_2 = 0$ (and even more so at any higher closing fee of auction house 2), and hence, setting the monopoly closing fee is the best reply for auction house 1.

Conversely, if a monopoly equilibrium exists, then after setting the monopoly closing fee, μ_1^M, auction house 1 attracts sellers of all types for any closing fee of auction house 2, including $\mu_2 = 0$. Therefore, $\mu_1^M \leq \phi_H(0)$. **End of Proof.**

The following notation will be useful later. Consider two extreme cases $\alpha = 1$ and $\alpha = 0$. In the first case, all sellers are L-type and auction house i solves the following maximization problem

$$\bar{w}_i^L = \max_{\mu_i} w_i^L(\mu_i) \cdot F_s(u_i^L(\mu_i)). \tag{11}$$

The unique maximizer of (11), μ_i^L, will be called the *L closing fee* of auction house i. In the second case, all sellers are H-type and auction house i solves

$$\bar{w}_i^H = \max_{\mu_i} w_i^H(\mu_i) \cdot F_s(u_i^H(\mu_i)). \tag{12}$$

The unique maximizer of (12), μ_i^H, will be called the *H closing fee* of auction house i.

5.2 Contestable Monopoly

Let $n_1 > n_2 > 0$. Consider an equilibrium where the more popular auction house 1 sets a positive closing fee less than the monopoly fee, attracts sellers of all types, and receives a positive expected payoff, while auction house 2 attracts no sellers and receives zero payoff. This situation is a *contestable monopoly*: the more popular auction house 1 is a monopolist who is forced to set the closing fee low enough (lower than the monopoly closing fee) to

keep the other auction house from "entering the market" (setting a closing fee above zero) and obtaining a positive expected payoff. In a *contestable monopoly equilibrium* the more popular auction house 1 sets $\mu_1 = \phi_H(0) < \mu_1^M$ and attracts both types of sellers, or $\lambda_2^\theta(\mu_1, \mu_2) = 0$ for each $\theta = H, L$.

Proposition 4 *Let $n_1 > n_2$. A contestable monopoly equilibrium exists if and only if and only if*

$$\mu_1^M > \phi_H(0) \tag{13}$$

and

$$\bar{w}_1(\phi_H(0), 0) \geq \bar{w}_1(\min\left\{\phi_L(0), \mu_1^L\right\}, 0). \tag{14}$$

Proof. Suppose that auction houses' closing fees are $(\phi_H(0), 0)$. Condition (13) means that H-type sellers prefer auction house 2 under the monopoly fee μ_1^M, hence this is not the monopoly equilibrium. Condition (14) mean that auction house 1 has no incentive to increase its closing fee to the level that would maximize the revenue from L-type sellers only (completely ignoring H-type sellers); clearly, auction house 1 cannot benefit by a reduction μ_1, and action house 2 cannot benefit by an increase of μ_2.

Conversely, suppose a contestable monopoly equilibrium exists. In such an equilibrium, to attract sellers of all types, auction house 1 sets its closing fee at most $\phi_H(0)$, and (13) must hold, otherwise auction house 1 could have benefited by setting closing fee μ_1^M and obtaining the monopoly equilibrium profit. Similarly, (14) must hold, otherwise auction house 1 could have benefited by setting closing fee μ_1^L. **End of Proof.**

Proposition 4 demonstrates that if auction house 1 is more popular than auction house 2 and cannot charge closing fee above $\phi_H(0)$ (because otherwise H-type and maybe even L-type sellers would switch to auction house 2) a contestable monopoly equilibrium arises with the closing fees $(\mu_1, \mu_2) = (\phi_H(0), 0)$ and $\lambda_2^\theta(\phi_H(0), 0) = 0$ for each $\theta = H, L$.

5.3 Market Segmentation

A contestable monopoly is a situation that is natural to see in our model where two different auction houses compete in prices for sellers. However, we show that this is not the only equilibrium outcome for competing auction houses.

Suppose that condition (14) does not hold. It means that auction house 1 can obtain a higher payoff if it charges a closing fee above $\phi_H(0)$ and attracts only L-type sellers, than if it charges closing fee equal to $\phi_H(0)$ and attracts all types of sellers. This can happen, for instance, when the mass of H-type sellers in the population is small enough. Now, auction house 2 can also raise its closing fee to collect a positive revenue from H-type sellers. Thus, the market is split into two segments where each auction house attracts one type of sellers and receives positive expected profit. An equilibrium where the L-type sellers prefer the more popular auction house 1 and the H-type sellers prefer the less popular auction house 2 will be called a *market segmentation equilibrium.*[10]

Proposition 5 *In a market segmentation equilibrium the following holds:*

(i) *Each auction house sets a monopoly fee on the respective market segment, i.e.,* $\mu_1 = \mu_1^L$ *and* $\mu_2 = \mu_2^H$*, and the sellers prefer the respective auction house,*

$$\phi_H(\mu_2^H) < \mu_1^L < \phi_L(\mu_2^H)$$

and

$$\phi_L^{-1}(\mu_1^L) < \mu_2^H < \phi_H^{-1}(\mu_1^L).$$

[10]Note that the opposite situation, where the L-type sellers prefer the less popular auction house 2 and the H-type sellers prefer the more popular auction house 1, is impossible by Proposition 2.

(ii) *A payoff of auction house j depends only on its own auction fees and is given by*

$$\bar{w}_1 = \alpha \max_{\mu_1} w_1^L(\mu_1) \cdot F_s(u_1^L(\mu_1))$$

and

$$\bar{w}_2 = (1-\alpha) \max_{\mu_2} w_2^H(\mu_2) \cdot F_s(u_2^H(\mu_2)).$$

Part (i) of the proposition shows that a market segmentation equilibrium can exist only inside area **B** (Figure 1); part (ii) shows that each auction house receives the monopoly payoff on the respective market segment. Note that since the monopoly fees μ_1^L and μ_2^H are unique, if a market segmentation equilibrium exists, it must be unique.

5.4 Characterization of Equilibria

The following theorem summarizes the above results in Section 5 and completes the characterization of equilibria.

Theorem 2 *There exists at most one equilibrium.*
If $n_1 = n_2$, then equilibrium closing fees are $(\mu_1, \mu_2) = (0, 0)$.
If $n_1 > n_2$, then the equilibrium is either monopoly, contestable monopoly, or market segmentation equilibrium.

Theorem 2 demonstrates that if an equilibrium exists, it must be one of the types we have discussed. Furthermore, it is unique[11] since each of the equilibrium types is uniquely defined. Note that since conditions for existence of the monopoly, contestable monopoly, and market segmentation equilibrium are mutually exclusive and do not cover the entire set of parameters, it follows that generally an equilibrium (in pure strategies) need not exist.

[11] Given our assumption that if an auction house receives zero profit in equilibrium, it sets zero fees.

6 Conclusion

We study competition between two auction houses that attract sellers by optimal choice of listing and closing fees. We characterize Bayesian Nash equilibria and show that there exists at most one equilibrium which, depending on the model parameters, may be one of the following types: Bertrand competition, monopoly, contestable monopoly, or market segmentation.

Our paper is an important complement to the existing literature on competing auction houses (Ellison, Fudenberg, and Möbius, 2004; Moldovanu, Sela, and Shi, 2008) where the result is that even though two auction houses can co-exist in equilibrium, the "law of one price" (i.e., the expected closing price is the same in both auction houses) should hold and the buyer/seller ratio should be constant across auction houses.

In a field experiment, Brown and Morgan (2009) demonstrate that eBay prices have been consistently higher than those on Yahoo, and eBay attracted more buyers per seller. In contrast to previous theoretical models, our model helps to explain this observation. It happens in the market segmentation equilibrium where highly popular eBay attracts impatient sellers and smaller auction houses attract more patient sellers.

Appendix

Proof of Lemma 1

Rewrite (1) as follows,

$$u_j(r_j, u^*) = -c_j + (1-\mu_j)\left[P_j(r_j)\left(x_j(r_j) - \frac{\delta_s u^*}{1-\mu_j}\right)\right] + \delta_s u^*.$$

Then, it is clear that

$$\arg\max_{r_j} u_j(r_j, u^*) = \arg\max_{r_j}\left[P_j(r_j)\left(x_j(r_j) - \frac{\delta_s u^*}{1-\mu_j}\right)\right].$$

The result follows from Lemma 5 below with $z = \frac{\delta_s u^*}{1-\mu_j}$.

Lemma 5 *For every $z \geq 0$, expression $P_j(r)(x_j(r) - z)$ is strictly quasiconcave in r. A unique solution r^* of the maximization problem*

$$\max_{r \in [\underline{v}, \bar{v}]} P_j(r)(x_j(r) - z)$$

is a unique solution of the following equation

$$r^* - \frac{1 - F(r^*)}{f(r^*)} = z, \text{ if } z < \bar{v} - \frac{1 - F(\bar{v})}{f(\bar{v})},$$

or

$$r^* = \bar{v}, \text{ if } z \geq \bar{v} - \frac{1 - F(\bar{v})}{f(\bar{v})}.$$

The result is standard in the literature (e.g., Krishna 2002, Ch. 2.5) and thus the proof is omitted.

Proof of Lemma 2

Equation (4) is equivalent to

$$\begin{aligned} u_j^* &= \max_r \left[-c_j + (1 - \mu_j) P_j(r) \left(x_j(r) - \frac{\delta_s u_j^*}{1 - \mu_j} \right) + \delta_s u_j^* \right] \\ &= -c_j + \delta_s u_j^* + (1 - \mu_j) \max_r \left[P_j(r) \left(x_j(r) - \frac{\delta_s u_j^*}{1 - \mu_j} \right) \right]. \end{aligned} \tag{15}$$

Note that the seller auctions off her object on auction house j only if

$$c_j < (1 - \mu_j) \max_r \left[P_j(r) x_j(r) \right]. \tag{16}$$

It means that $\mu_j < 1$. After dividing both sides by $(1 - \mu_j)$ and rearranging the terms in (15), we obtain

$$\frac{c_j + (1 - \delta_s) u_j^*}{(1 - \mu_j)} = \max_r \left[P_j(r) \left(x_j(r) - \frac{\delta_s u_j^*}{1 - \mu_j} \right) \right]. \tag{17}$$

Note that from (16) the left-hand side is smaller (greater) than the right-hand side at $u_j^* = 0$ ($u_j^* = 1$). Since the left-hand side is increasing and the right-hand side is decreasing in u_j^* (recall that $P_j(\cdot)$ and $x_j(\cdot)$ are nonnegative), equation (17) has a unique solution. **End of Proof**.

Proof of Theorem 1

Consider arbitrary strategies of auction houses $a_1 = (c_1, \mu_1)$ and $a_2 = (c_2, \mu_2)$. Suppose that $c_i > 0$ and $\bar{w}_i(a_i, a_j) > 0$, $i = 1$ or 2. Since $\bar{w}_i(a_i, a_j) > 0$, auction house i attracts a positive measure of sellers, $\lambda_i^H(a_i, a_j) + \lambda_i^L(a_i, a_j) > 0$. Suppose that strategy a_i is auction house i's best reply to the strategy a_j, that is, for every strategy $a_i' = (c_i', \mu_i')$, $\bar{w}_i(a_i, a_j) \geq \bar{w}_i(a_i', a_j)$.

Recall that the payoff of θ-type seller, $\theta = H, L$, is given by

$$u_i^\theta = -c_i + (1 - \mu_i)P_i(r^\theta)x_i(r^\theta) + (1 - P_i(r^\theta))\delta_\theta u_i^\theta, \tag{18}$$

where r^θ from Lemma 1 is given by

$$r^\theta - \frac{1 - F(r^\theta)}{f(r^\theta)} = \frac{\delta_\theta u^\theta}{1 - \mu_i}. \tag{19}$$

The payoff of auction house i conditional on interaction with θ-type seller, is given by

$$w_i^\theta = c_i + \mu_i P_i(r^\theta)x_i(r^\theta) + (1 - P_i(r^\theta))\gamma w_i^\theta. \tag{20}$$

The sum of the two payoffs is equal to

$$W_i^\theta(r^\theta) \equiv u_i^\theta + w_i^\theta = P_i(r^\theta)x_i(r^\theta) + (1 - P_i(r^\theta))(\delta_\theta u_i^\theta + \gamma w_i^\theta). \tag{21}$$

Note that this sum does not depend on the auction house i's fees directly, only via the seller's choice of the reserve price r^θ. We will show now that, whenever $c_i > 0$, the sum of the seller's and auction house's payoffs is strictly increasing in r in a small neighborhood of r^θ. Further, we will show that there exists a change of fees, an increase in μ_i and a simultaneous decrease in c_i

that shifts up the reserve price, thus increasing in the sum of the payoffs and making auction house i strictly better off.

Lemma 6 *If $c_i > 0$, then for each type $\theta = H, L$ there exists a neighborhood of r^θ where $W_i^\theta(\cdot)$ is strictly increasing.*

Proof. After rearranging the terms of (21), we obtain

$$W_i^\theta(r) = P_i(r)(x_i(r) - (\delta_\theta u_i^\theta + \gamma w_i^\theta)) + \delta_\theta u_i^\theta + \gamma w_i^\theta.$$

Let r^* be the reserve price that maximizes $W_i^\theta(r)$. By Lemma 5, it is given by

$$r^* - \frac{1 - F(r^*)}{f(r^*)} = \delta_\theta u_i^\theta + \gamma w_i^\theta, \tag{22}$$

and, furthermore, $W_i^\theta(r)$ is strictly quasiconcave. Hence $W_i^\theta(r)$ is strictly increasing in r whenever $r < r^*$.

Solving (18) for u_i^θ, we obtain

$$u_i^\theta = \frac{(1-\mu_i)P_i(r^\theta)x_i(r^\theta) - c_i}{1 - \delta_\theta(1 - P_i(r^\theta))}.$$

Solving (20) for w_i^θ, we get

$$w_i^\theta = \frac{\mu_i P_i(r^\theta)x_i(r^\theta) + c_i}{1 - \gamma(1 - P_i(r^\theta))}.$$

Now, using the assumption that $\gamma \geq \delta_\theta$, we obtain

$$\begin{aligned}
\delta_\theta u_i^\theta + \gamma w_i^\theta &\geq \delta_\theta \frac{(1-\mu_i)P_i(r^\theta)x_i(r^\theta) - c_i}{1 - \delta_\theta(1 - P_i(r^\theta))} + \delta_\theta \frac{\mu_i P_i(r^\theta)x_i(r^\theta) + c_i}{1 - \gamma(1 - P_i(r^\theta))} \\
&\geq \delta_\theta \frac{P_i(r^\theta)x_i(r^\theta)}{1 - \delta_\theta(1 - P_i(r^\theta))} = \frac{\delta_\theta}{(1-\mu_i)} \frac{(1-\mu_i)P_i(r^\theta)x_i(r^\theta)}{1 - \delta_\theta(1 - P_i(r^\theta))} \\
&\geq \frac{\delta_\theta}{(1-\mu_i)} \frac{(1-\mu_i)P_i(r^\theta)x_i(r^\theta) - c_i}{1 - \delta_\theta(1 - P_i(r^\theta))} \equiv \frac{\delta_\theta}{(1-\mu_i)} u_i^\theta.
\end{aligned}$$

The last inequality is due to the assumption that $c_i \geq 0$ and it is strict whenever $c > 0$. Therefore, $\delta_\theta u_i^\theta + \gamma w_i^\theta > \frac{\delta_\theta u_i^\theta}{1-\mu_i}$ whenever $c_i > 0$.

Since $z - \frac{1-F(z)}{f(z)}$ is strictly increasing, it follows that $r^\theta < r^*$ from (19) and (22). Hence, $W(r)$ is strictly increasing in some neighborhood of r^θ. **End of Proof**.

We continue with the proof of Theorem 1. Consider a hypothetical situation where auction house i can choose a different pair of fees, a_i', only for the current period, while keeping the original fees, a_i, in the next and further periods (thus affecting only payoffs from this period, but not continuation payoffs). By the one-period deviation principle, if no choice of a_i' in the current period can lead to an increase in i's profit, neither can any change of fees in all periods.

Let μ_i' be a closing fee in a small neighborhood of μ_i, $\mu_i' > \mu_i$. Since the fees will be changed only in the current period, the continuation payoff of a θ-type seller, u_i^θ, and the continuation payoff of auction house i conditional on the interaction with θ-type seller, w_i^θ, remain unchanged. By (19) and by the hazard rate assumption, the reserve price under the new closing fee, r_θ', is strictly greater than the original reserve price, r_θ, and by Lemma 6, $W_i^\theta(r_\theta') > W_i^\theta(r_\theta)$. Let ε be a small positive number such that

$$0 < \varepsilon < W_i^\theta(r_\theta') - W_i^\theta(r_\theta),\ \theta = H, L.$$

Consider fees (c_i', μ_i') such that the listing fee $0 \leq c_i' < c_i$ and such that the expected payoff of auction house i is increased by ε in comparison with fees (c_i, μ_i). Since the total payoff increase is $W_i^\theta(r_\theta') - W_i^\theta(r_\theta)$, it follows that the seller's payoff must also increase. Note that, since only current-period fees are changed, the payoffs of auction house i and the seller are linear in c_i' (it is a simple redistribution of the revenue). Hence, for every measures of H-type and L-type sellers, $\lambda_i^H(a_i, a_j)$ and $\lambda_i^L(a_i, a_j)$, the auction house can guarantee to obtain at least ε more revenue, which is a contradiction that the initial fees (c_i, μ_i) are optimal. **End of Proof**.

Proof of Proposition 1

Proof. *Part (i).* Suppose that $n_1 = n_2$. Then a seller of any type prefers the auction house with a lower closing fee. If, say, $\mu_1 > \mu_2$, then auction house 2 attracts sellers of both types, H-type and L-type, and it can profitably deviate by setting a slightly higher closing fee (but still below μ_1). If $\mu_1 = \mu_2 > 0$, then the sellers are indifferent between the auction houses, and an auction house that attracts not all sellers can profitably deviate by setting a slightly lower closing fee.

Part (ii). Suppose that $n_1 > n_2$ and $\mu_1 = 0$. Then a seller of any type strictly prefers auction house 1 (even when $\mu_2 = 0$), and thus auction house 1 can profitably deviate by setting a slightly higher fee. **End of Proof**.

Proof of Proposition 2

We need to show that if the fees $(\mu_1, \mu_2) \neq (1, 1)$ are such that $u_1^H = u_2^H$, then $u_1^L > u_2^L$. Denote

$$\tilde{u}_j^\theta = \frac{u_j^\theta}{1 - \mu_j}, \ j = 1, 2 \text{ and } \theta = H, L.$$

Then, $u_j^\theta = (1 - \mu_j)\tilde{u}_j^\theta$, and thus $u_1^H = u_2^H$ implies $u_1^L > u_2^L$ if and only if

$$\frac{1 - \mu_2}{1 - \mu_1} = \frac{\tilde{u}_1^H}{\tilde{u}_2^H} < \frac{\tilde{u}_1^L}{\tilde{u}_2^L}. \tag{23}$$

Dividing both sides of (1) by $1 - \mu_j$ and maximizing the left-hand side w.r.t. r (with $c_j = 0$), we have

$$\tilde{u}_j^\theta = \max_r \left[P_j(r) x_j(r) + (1 - P_j(r)) \delta_\theta \tilde{u}_j^\theta \right].$$

It follows from Lemma 2 that the above equation has a unique solution $\tilde{u}_j^\delta$. Note that $\tilde{u}_j^\theta$ does not depend on μ_j, and neither does the optimal reserve price r_j^θ. Also, since $n_1 > n_2$, we have $P_1(r) > P_2(r)$ and $x_1(r) > x_2(r)$ for every r, and it is straightforward to show that $\tilde{u}_1^\theta > \tilde{u}_2^\theta$ and $r_1^\theta > r_2^\theta$, $\theta = H, L$.

Define

$$Q(n_j, z) = \max_r \left[P_j(r) x_j(r) + (1 - P_j(r)) z \right]. \tag{24}$$

Thus we have $\tilde{u}_j^\theta = Q(n_j, \delta_\theta \tilde{u}_j^\theta)$. Note that function $Q(n_j, z)$ satisfies the submodularity condition in (n_j, z):

$$\frac{Q(n_1, z_1)}{Q(n_2, z_1)} < \frac{Q(n_1, z_2)}{Q(n_2, z_2)}$$

whenever $n_1 > n_2$ and $z_1 > z_2$. To see this, take the partial derivative of Q with respect to z. By the Envelope Theorem, it is equal to $1 - P_j(r)$, where r is the maximizer of (24). Since $P_j(r)$ is strictly increasing in the number of bidders (see Proof of Lemma 5), the submodularity of Q is immediate. Let $z_1 = \delta_H \tilde{u}_1^H$ and $z_2 = \delta_L \tilde{u}_2^L$. Using $\tilde{u}_1^H > \tilde{u}_1^L > \tilde{u}_2^L$ and $\tilde{u}_1^H > \tilde{u}_2^H > \tilde{u}_2^L$, we obtain inequality (23). **End of Proof**.

Proof of Lemma 4

Solving (18) for u_i^θ (with $c_i = 0$), we obtain

$$u_i^\theta(\mu_i) = \frac{(1 - \mu_i) P_i(r^\theta) x_i(r^\theta)}{1 - \delta_\theta (1 - P_i(r^\theta))}.$$

After dividing both sides by $1 - \mu_i$, we have

$$\tilde{u}_j^\theta \equiv \frac{u_i^\theta(\mu_i)}{1 - \mu_i} = \frac{P_i(r^\theta) x_i(r^\theta)}{1 - \delta_\theta (1 - P_i(r^\theta))}.$$

By the argument provided in Proof of Proposition 2, we know that $\tilde{u}_j$ and r^θ are independent of μ_i, and hence $u_i^\theta(\mu_i) = (1 - \mu_i)\tilde{u}_j^\theta$ is linear in μ_i. Also, solving (20) for w_i^θ, we obtain

$$w_i^\theta(\mu_i) = \frac{\mu_i P_i(r^\theta) x_i(r^\theta)}{1 - \gamma (1 - P_i(r^\theta))},$$

i.e., $w_i^\theta(\mu_i) = \mu_i \tilde{w}_i^\theta$, where $\tilde{w}_i^\theta$ is a constant w.r.t. μ_i. Thus,

$$w_i^\theta(\mu_i) \cdot F_s(u_i^\theta(\mu_i)) = \mu_i \tilde{w}_i^\theta \cdot F_s((1 - \mu_i)\tilde{u}_j^\theta).$$

Taking the derivative with respect to μ_i and denoting $z = (1-\mu_i)\tilde{u}_j^\theta$, we obtain

$$\begin{aligned}\tilde{w}_i^\theta \cdot F_s(z) - \mu_i \tilde{w}_i^\theta \cdot \tilde{u}_j^\theta f_s(z) &= \tilde{w}_i^\theta \left(F_s(z) - \mu_i \tilde{u}_j^\theta f_s(z)\right) \\ &= \tilde{w}_i^\theta \left(F_s(z) - (\tilde{u}_j^\theta - z) f_s(z)\right).\end{aligned}$$

Since by assumption on F_s (see Section 2) expression $[F_s(z) - (\bar{v} - z) f_s(z)]$ is strictly increasing in z, its derivative satisfies

$$2f_s(z) - (\bar{v} - z) f_s'(z) > 0$$

and, furthermore, this inequality holds for every $z \leq \tilde{u}_j^\theta$ even after we replace $\bar{v}$ by $\tilde{u}_j^\theta$ (as $\tilde{u}_j^\theta \leq \bar{v}$). Thus $F_s(z) - (\tilde{u}_j^\theta - z) f_s(z)$ is also strictly increasing in $z \leq \tilde{u}_j^\theta$. Since z is strictly decreasing in μ_i, it follows that $\mu_i \tilde{w}_i^\theta \cdot F_s((1-\mu_i)\tilde{u}_j^\theta)$ is strictly concave.

Proof of Proposition 5

Proof. To prove part (i), we need to show that the reverse segmentation, where L-type sellers prefer auction house 2 and H-type sellers prefer auction house 1, cannot occur in equilibrium. This is immediate by Proposition 2, according to which, for every pair of fees, H-type sellers prefer auction house 1 only if L-type sellers also prefer auction house 1.

Part (ii). Consider, say, auction house 1. By part (i), the best-reply fee is an interior solution of the problem of finding the best fee facing the population of L-type sellers only. But this solution is equal to the unique auction fee on the monopoly market with only L-type sellers, μ_1^L. **End of Proof**.

Proof of Theorem 2

Equilibria in the case of $n_1 = n_2$ are fully characterized by Proposition 1.

Suppose that $n_1 > n_2$. We described three types of equilibria, monopoly, contestable monopoly, and market segmentation equilibrium and showed that if an equilibrium exists, it is unique. It remains to show that no other equilibria may exist.

Consider an equilibrium $s = ((\mu_1, \mu_2), (\lambda_1^H, \lambda_2^H, r_1^H, r_2^H), (\lambda_1^L, \lambda_2^L, r_1^L, r_2^L))$. First, assume that $\lambda_j^\theta \in \{0, 1\}$ for every $j = 1, 2$ and every $\theta = H, L$. Note that s is a monopoly or contestable monopoly equilibrium if $\lambda_2^H = \lambda_2^L = 0$. Since $n_1 > n_2$, clearly, $\lambda_1^H = \lambda_1^L = 0$ cannot occur in equilibrium, as auction house 1 can charge low enough closing fee to attract sellers (see Figure 1). Next, note that if $\lambda_1^H = 0$ and $\lambda_2^L = 0$, s is a market segmentation equilibrium, and $\lambda_2^H = 0$ and $\lambda_1^L = 0$ cannot occur in equilibrium by Proposition 2.

Finally, suppose that $0 < \lambda_j^\theta < 1$ for some $j = 1, 2$ and some $\theta = H, L$, that is, a θ-type seller is indifferent between two auction houses (recall our assumption that if the seller is indifferent between auctioning the object or consuming it, he consumes with probability 1). Then s cannot be an equilibrium: by Proposition 1 (ii), at least one auction house receives positive profit and thus it can attract θ-type sellers with probability one by marginally reducing its closing fee. **End of Proof.**

References

Bikhchandani, S. and C-f. Huang (1989). "Auctions with Resale Markets: An Exploratory Model of Treasury Bill Markets." *Review of Financial Studies* **2**, 311–339.

Bose, S. and G. Deltas (1999). "Welcoming the Middlemen: Restricting Competition in Auctions by Excluding the Consumers." In *Current Trends in Economics: Theory and Applications*, Studies in Economic Theory, vol. 8 (Aliprantis, Alkan, and Yannelis, Eds.), 119–131. Springer-Verlag.

Bose, S. and G. Deltas (2007). "Exclusive vs. Non-Exclusive Dealing in Auctions with Resale." *Economic Theory* **31**, 1–17.

Brown, J. and J. Morgan (2009). "How Much Is a Dollar Worth? Tipping versus Equilibrium Coexistence on Competing Online Auction Sites." *Journal of Political Economy* **117**, 668–700.

Calzolari, G. and A. Pavan (2006). "Monopoly with Resale." *RAND Journal of Economics* **37**, 362–375.

Ellison, G.,D. Fudenberg and M. Möbius (2004). "Competing Auctions." *Journal of the European Economic Association* **2**, 30–66.

Fudenberg, D., D. Levin and J. Tirole (1985). "Infinite-Horizon Models of Bargaining with One-Sided Incomplete Information." in *Game Theoretic Models of Bargaining* (A. Roth, Ed.), 73–98. Cambridge University Press.

Garratt, R. and T. Tröger (2006). "Speculation in Standard Auctions with Resale." *Econometrica* **74**, 753–769.

Gupta, M. and B. Lebrun (1999). "First price auctions with resale." *Economics Letters* **64**, 181–185.

Haile, P. (1999). "Auctions with Resale." Working paper, University of Wisconsin-Madison.

Haile, P. (2000). "Partial pooling at the reserve price in auctions with resale opportunities." *Games and Economic Behavior* **33**, 231–248.

Haile, P. (2001). "Auctions with Resale Markets: An Application to U.S. Forest Service Timber Sales." *American Economic Review* **91**, 399-427.

Haile, P. (2003). "Auctions with private uncertainty and resale opportunities." *Journal of Economic Theory* **108**, 72–110.

Horstmann, I. and C. LaCasse (1997). "Secret Reserve Prices in a Bidding Model with a Resale Option." *American Economic Review* **87**, 663-684.

Krishna, V. (2002). *Auction Theory.* Academic Press.

Matros, A. and A. Zapechelnyuk (2008). Optimal Fees in Internet Auctions. *Review of Economic Design* **12**, 155–163.

McAfee, P. and D. Vincent (1997). Sequentially optimal auctions. *Games and Economic Behavior* **18**, 246–276.

Milgrom, P. (1987). "Auction theory." In *Advances in Economic Theory: Fifth World Congress* (T. Bewley, Ed.), 1–32. Cambridge University Press.

Moldovanu, B., A. Sela and X. Shi (2008). "Competing Auctions with Endogenous Quantities." *Journal of Economic Theory* **141**, 1–27.

Myerson, R. (1981). "Optimal Auction Design." *Mathematics of Operations Research* **6**, 58–73.

Pagnozzi, M. (2007). "Bidding to Lose? Auctions with Resale." *RAND Journal of Economics* **38**, 1090–1112.

Zheng, C. (2002). "Optimal Auction with Resale." *Econometrica* **70**, 2197–2224.