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Mixed Strategy Nash equilibria**

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On the Existence of Mixed Strategy Nash Equilibria

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Abstract The focus of this paper is on developing easily verifiable sufficient conditions for the existence of a mixed strategy Nash equilibrium for both diagonally transfer continuous and better-reply secure games. First, we show that employing the concept of diagonal transfer continuity in place of better-reply security might be advantageous when the existence of a mixed strategy Nash equilibrium is concerned. Then, we study equilibrium existence in better-reply secure games possessing a payoff secure mixed extension. With the aid of an example we show that such games need not have mixed strategy Nash equilibria. We provide geometric conditions for the mixed extension of a two-person game that is reciprocally upper semicontinuous and uniformly payoff secure to be better-reply secure.

Keywords Discontinuous game; Diagonally transfer continuous game; Better-reply secure game; Mixed strategy equilibrium; Transfer lower semicontinuity

JEL classification Numbers C65; C72

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1 Introduction

Nowadays, one of the main tools in the arsenal of economists concerned with equilibrium existence is Reny's (1999) theorem, according to which a compact Borel game has a mixed strategy Nash equilibrium if its mixed extension is better-reply secure.³ In applications, better-reply security usually follows from two conditions: one related to reciprocal upper semicontinuity and the other to payoff security.

Establishing the payoff security of a game's mixed extension often constitutes a complicated problem. The concept of uniform payoff security, introduced by Monteiro and Page (2007), makes the problem considerably more tractable in games where it is applicable, including catalog games (Page and Monteiro, 2003) and voting models (Carbonell-Nicolau and Ok, 2007).⁴ Verifying whether a game that is not upper semicontinuous-sum has a better-reply secure mixed extension is, as a rule, quite challenging. Our main focus is on studying the existence of a mixed strategy Nash equilibrium in normal form games where the sum of the payoff functions is not necessarily upper semicontinuous.

We begin with considering the games having diagonally transfer continuous mixed extensions. Baye, Tian, and Zhou (1993) showed that the existence of a pure strategy Nash equilibrium in diagonally transfer continuous games follows from a generalization of the Knaster–Kuratowski–Mazurkiewicz (KKM) lemma.⁵ In Section 2 of this paper, the Ky Fan min-max inequality, in a slightly generalized form, is used to prove that every compact Borel game whose mixed extension is diagonally transfer continuous

³A number of results extending Reny's equilibrium existence theorem have been obtained recently (see, e.g., Barelli and Meneghel, 2013; Bich, 2009; Carmona, 2011; de Castro, 2011; McLennan, Monteiro, and Tourky, 2011)

⁴Another approach to showing the payoff security of mixed extensions can be found in Duggan (2007), where hospitable strategies are used to study equilibrium existence in voting models.

⁵The first part of this paper contains a number of results first presented in our 2012 working paper "On Uniform Conditions for the Existence of Mixed Strategy Equilibria."

has a mixed strategy Nash equilibrium. The range of applications of this basic result is considerably broader than that of Glicksberg’s (1952) equilibrium existence theorem – whose proof is based on the Kakutani-Fan-Glicksberg fixed point theorem. In particular, the mixed extension of a game is diagonally transfer continuous if the following two conventional assumptions hold: the extension is payoff secure and the game is upper semicontinuous-sum. Then, in Section 3, we extend the concept of uniform payoff security to diagonally transfer continuous games by introducing uniform diagonal security. In the upper semicontinuous-sum games, these two uniform conditions coincide. At the same time, if a compact Borel game is uniformly diagonally secure, it has a mixed strategy Nash equilibrium, which makes it possible to avoid having to study any additional properties of the game’s mixed extension.

Example 1 is a slight modification of the Tullock rent-seeking game where it is additionally assumed that the favor the players vie for is granted to a third party with probability one-half if at least one player exerts no effort at all. Notwithstanding the fact that the game is not better-reply secure, it is not only diagonally transfer continuous, but also uniformly diagonally secure; that is, the game has a mixed strategy Nash equilibrium.

In Section 4 we adapt Simon’s (1987, p. 577) concept of weak domination on average to our setting by introducing weak uniform payoff security, a generalization of uniform payoff security. Using this concept, we construct a better-reply secure two-person game with a payoff secure mixed extension that has no mixed strategy equilibrium (Example 2).

In Section 5, we study the existence of a mixed strategy equilibrium in reciprocally upper semicontinuous games that are uniformly payoff secure. In such games, the equilibrium existence problem becomes considerably more tractable if it is possible to transform the game into an upper semicontinuous-sum game with the aid of positive affine transformations,⁶ which, in particu-

⁶Using a similar approach, Amir (2005) gives examples of Cournot oligopolies possessing the cardinal complementarity property where the other complementarity conditions are ineffective.

lar, implies that the game also has a reciprocally upper semicontinuous mixed extension. Example 3 applies this technique to a conventional two-candidate probabilistic spatial voting game. Theorems 5 and 6 give geometric sufficient conditions for games that are reciprocally upper semicontinuous and uniformly payoff secure to have a better-reply secure mixed extension.

The Appendix contains a number of auxiliary results, deferred proofs, and some comments regarding Theorem 5b of Dasgupta and Maskin (1986).

2 The Model and Some Facts

We consider a game $G = (X_i, u_i)_{i \in I}$, where $I = \{1, \dots, n\}$, each player i 's pure strategy set X_i is a nonempty, compact subset of a metrizable topological vector space, and each payoff function u_i is a bounded Borel measurable function from the Cartesian product $X = \prod_{i \in I} X_i$, equipped with the product topology, to \mathbb{R} . Under these conditions, $G = (X_i, u_i)_{i \in I}$ is called a compact Borel game. In this paper, by a game we mean a compact Borel game. A game $G = (X_i, u_i)_{i \in I}$ is quasiconcave if each X_i is convex and $u_i(\cdot, x_{-i}) : X_i \rightarrow \mathbb{R}$ is quasiconcave for all $i \in I$ and all $x_{-i} \in X_{-i}$, where $X_{-i} = \prod_{k \in I \setminus \{i\}} X_k$.

The following definition of a payoff secure game was given by Reny (1999).

Definition 1 *In $G = (X_i, u_i)_{i \in I}$, player i can secure a payoff of $\alpha \in \mathbb{R}$ at $x \in X$ if there exists $d_i \in X_i$ such that $u_i(d_i, x'_{-i}) \geq \alpha$ for all x'_{-i} in some open neighborhood of x_{-i} . The game G is payoff secure if for every $x \in X$ and every $\varepsilon > 0$, each player i can secure a payoff of $u_i(x) - \varepsilon$ at x .*

Payoff security can be reformulated in terms of transfer lower semicontinuity, due to Tian (1992).

Definition 2 *Let Z and Y be two topological spaces. A function $f : Z \times Y \rightarrow \mathbb{R}$ is λ -transfer lower semicontinuous in y if, for every $(z, y) \in Z \times Y$,*

$f(z, y) > \lambda$ implies that there exists some point $z' \in Z$ and some neighborhood $\mathcal{N}_Y(y)$ of y in Y such that $f(z', w) > \lambda$ for all $w \in \mathcal{N}_Y(y)$. A function $f : Z \times Y \rightarrow \mathbb{R}$ is transfer lower semicontinuous in y if f is λ -transfer lower semicontinuous in y for every $\lambda \in \mathbb{R}$.

A game is payoff secure if and only if each player's payoff function is transfer lower semicontinuous in the other players' strategies (see Prokopovych, 2011, Lemma 1).

The graph of G is defined by $\text{Gr}G = \{(x, u) \in X \times \mathbb{R}^n \mid u_i(x) = u_i \text{ for all } i \in N\}$, and the set of pure strategy Nash equilibria of G in X is denoted by E_G . For a subset B of a topological vector space X , we denote by $\text{cl}B$ the closure of B and by $\text{co}B$ the convex hull of B . In a metric space Y , we denote by $B_Y(y, r)$ the open ball centered at y and with radius $r > 0$.

Definition 3 A game $G = (X_i, u_i)_{i \in I}$ is better-reply secure if whenever $(x^*, u^*) \in \text{clGr}G$ and $x^* \in X \setminus E_G$, some player i can secure a payoff strictly above u_i^* at x^* .

A useful fact is that a payoff secure game is better-reply secure iff it is also transfer reciprocally upper semicontinuous (see Bagh and Jofre, 2006; Prokopovych, 2011, Lemma 2).

Definition 4 A game $G = (X_i, u_i)_{i \in I}$ is: (i) reciprocally upper semicontinuous if for any $(x, \alpha) \in \text{clGr}G \setminus \text{Gr}G$, there is a player i such that $u_i(x) > \alpha_i$; (ii) weakly reciprocally upper semicontinuous if whenever $(x, \alpha) \in \text{clGr}G \setminus \text{Gr}G$, there are a player i and $d_i \in X_i$ such that $u_i(d_i, x_{-i}) > \alpha_i$; (iii) transfer reciprocally upper semicontinuous if whenever $(x, \alpha) \in \text{clGr}G \setminus \text{Gr}G$ and x is not a Nash equilibrium, there are a player i and $d_i \in X_i$ such that $u_i(d_i, x_{-i}) > \alpha_i$.

It is clear that every weakly reciprocally upper semicontinuous game is transfer reciprocally upper semicontinuous.

Reny's (1999) equilibrium existence theorem states that every compact, quasiconcave, better-reply secure game has a Nash equilibrium in pure strategies.

Theorem 1 (Reny 1999) *If $G = (X_i, u_i)_{i \in I}$ is compact, quasiconcave, and better-reply secure, then it possesses a pure strategy Nash equilibrium.*

Another approach to studying equilibrium existence in discontinuous games was proposed by Baye, Tian, and Zhou (1993).

For $G = (X_i, u_i)_{i \in I}$, define the following aggregator functions:

$$A_G : X \times X \rightarrow \mathbb{R} \text{ by } A_G(d, x) = \sum_{i \in I} u_i(d_i, x_{-i}),$$

where, as usual, the $-i$ subscript on x stands for "all players except i ,"

$$A_G^0 : X \rightarrow \mathbb{R} \text{ by } A_G^0(x) = \sum_{i \in I} u_i(x),$$

and

$$F_G : X \times X \rightarrow \mathbb{R} \text{ by } F_G(d, x) = A_G(d, x) - A_G^0(x).$$

A strategy profile $x \in X$ is a Nash equilibrium of G iff $F_G(d, x) \leq 0$ for all $d \in X$.

Definition 5 *A game $G = (X_i, u_i)_{i \in I}$ is diagonally transfer continuous if for every $x \in X \setminus E_G$, there exist some $d \in X$ and some neighborhood $\mathcal{N}_X(x)$ of x in X such that $F_G(d, z) > 0$ for all $z \in \mathcal{N}_X(x)$.*

It is worth noticing that G is diagonally transfer continuous iff F_G is 0-transfer lower semicontinuous in x .

Every payoff secure game with an upper semicontinuous A_G^0 is diagonally transfer continuous.

Lemma 1 *If, in a game $G = (X_i, u_i)_{i \in I}$, each $u_i : X \rightarrow \mathbb{R}$ is transfer lower semicontinuous in x_{-i} and the aggregator function $A_G^0 : X \rightarrow \mathbb{R}$ is upper semicontinuous, then G is diagonally transfer continuous.*

For convenience, the proof of Lemma 1 is given in the Appendix.

Now we define the mixed extension $\Gamma(G)$ of a game $G = (X_i, u_i)_{i \in I}$. Denote by $\Delta(X_i)$ the set of Borel probability measures on X_i and by $ca(X_i)$ the set of Borel signed measures with finite total variation on X_i . A basic open neighborhood of $\mu_i \in ca(X_i)$ in the weak topology on $ca(X_i)$ is a set of the form $\left\{ \nu_i \in ca(X_i) : \left| \int f_j(d\nu_i - d\mu_i) \right| < \varepsilon, j = 1, \dots, m \right\}$ for some continuous $f_j : X_i \rightarrow \mathbb{R}$, $j = 1, \dots, m$, and $\varepsilon > 0$. The set $ca(X_i)$ is a Hausdorff topological vector space equipped with the weak topology. The topology induced on $\Delta(X_i)$ by the weak topology is compact.⁷ Let each of the Cartesian products $ca(X) = ca(X_1) \times \dots \times ca(X_n)$ and $\Delta(X) = \Delta(X_1) \times \dots \times \Delta(X_n)$ be equipped with the product topology. The set $ca(X)$ is a Hausdorff topological vector space in which the operations of addition and scalar multiplication are defined as follows: for $\mu = (\mu_1, \dots, \mu_n) \in ca(X)$ and $\alpha \in \mathbb{R}$ the scalar multiplication of μ by α is the element $\alpha\mu$ given by $\alpha\mu = (\alpha\mu_1, \dots, \alpha\mu_n)$. The addition of $\mu = (\mu_1, \dots, \mu_n) \in ca(X)$ and $v = (\nu_1, \dots, \nu_n) \in ca(X)$ gives $\mu + v = (\mu_1 + \nu_1, \dots, \mu_n + \nu_n)$.

The mixed extension of the game G is the n -player normal form game $\Gamma(G) = (\Delta(X_i), U_i)_{i \in I}$, where $\Delta(X_i)$ is player i 's strategy set and player i 's payoff function $U_i : \Delta(X) \rightarrow \mathbb{R}$ is defined by

$$U_i(\mu) = \int_{X_1} \int_{X_2} \dots \int_{X_n} u_i(x_1, \dots, x_n) d\mu_1 \dots d\mu_n.$$

For the game $\Gamma(G)$, we also define the aggregator functions $A_{\Gamma(G)} : \Delta(X) \times \Delta(X) \rightarrow \mathbb{R}$, $A_{\Gamma(G)}^0 : \Delta(X) \rightarrow \mathbb{R}$, and $F_{\Gamma(G)} : \Delta(X) \times \Delta(X) \rightarrow \mathbb{R}$ (see the corresponding definitions for G).

Theorem 2 *If the mixed extension $\Gamma(G)$ of a game $G = (X_i, u_i)_{i \in I}$ is diagonally transfer continuous, then G possesses a mixed strategy Nash equilibrium.*

⁷In order to make $\Delta(X_i)$ a subset of a linear space, we embed it in the space $ca(X_i)$ of signed measures with finite total variation on X_i . Sometimes it is possible to proceed without the embedding. See, for example, the proof of the compactness of the set of probability measures given by Glycopantis and Muir (2004).

rium.

Proof. The set $\Delta(X)$ is a compact, convex subset of $ca(X)$. Consider the aggregator function $F_{\Gamma(G)}(\sigma, \mu) : \Delta(X) \times \Delta(X) \rightarrow \mathbb{R}$. Since $F_{\Gamma(G)}$ is linear in σ and 0-transfer lower semicontinuous in μ , the mixed extension $\Gamma(G)$ of G has a Nash equilibrium in pure strategies by the Ky Fan minimax inequality (see Lemma 4 in the Appendix). ■

Another proof of Theorem 2 can be obtained by using the fact that every diagonally transfer continuous mixed extension has the single deviation property (see Reny, 2009; Reny, 2011; and Prokopovych, 2013 for some details).

Since the upper semicontinuity of A_G^0 implies the upper semicontinuity of $A_{\Gamma(G)}^0$, verifying whether a game has a mixed strategy Nash equilibrium usually means verifying the following two properties: (a) the upper semicontinuity of the sum of the payoff functions; and (b) the payoff security of its mixed extension. If these properties hold, the mixed extension of the game is not only better-reply secure but, by Lemma 1, diagonally transfer continuous. For example, this is the case for the all-pay auction games with homogeneous valuations (see Baye, Kovenock, and Vries, 1996; Monteiro and Page, 2007).

The diagonal transfer continuity of a game does not imply that its mixed extension is diagonally transfer continuous. For example, Sion and Wolfe's (1957) zero-sum game is payoff secure (see Carmona, 2005) and its aggregator function A_G^0 is constant. Thus, the game is diagonally transfer continuous by Lemma 1. However, since the game has no mixed strategy Nash equilibria, its mixed extension is not diagonally transfer continuous.

3 Uniform Diagonal Security

An easily verifiable condition for the mixed extension of a game to be payoff secure is that of uniform payoff security, due to Monteiro and Page (2007).

Definition 6 A game $G = (X_i, u_i)_{i \in I}$ is uniformly payoff secure if, for each $i \in I$, every $x_i \in X_i$ and every $\varepsilon > 0$, there is $d_i(x_i, \varepsilon) \in X_i$ such that for every $x_{-i} \in X_{-i}$, $u_i(d_i(x_i, \varepsilon), w_{-i}) \geq u_i(x_i, x_{-i}) - \varepsilon$ for all w_{-i} in some open neighborhood $\mathcal{N}_{X_{-i}}(x_{-i})$ of x_{-i} in X_{-i} .

We will omit the arguments x_i and ε of d_i if there is no ambiguity. Another useful definition is the following: In $G = (X_i, u_i)_{i \in I}$, player i 's payoff function $u_i : X_i \times X_{-i} \rightarrow \mathbb{R}$ is uniformly transfer lower semicontinuous in x_{-i} at $x_i \in X_i$ if, for every $\varepsilon > 0$, there is $d_i \in X_i$ such that, for every $x_{-i} \in X_{-i}$, there exists a neighborhood $\mathcal{N}_{X_{-i}}(x_{-i})$ of x_{-i} in X_{-i} such that $u_i(d_i, w_{-i}) > u_i(x_i, x_{-i}) - \varepsilon$ for all $w_{-i} \in \mathcal{N}_{X_{-i}}(x_{-i})$.

The mixed extension of a uniformly payoff secure, upper semicontinuous-sum game is both better-reply secure and diagonally transfer continuous.

Corollary 1 If $G = (X_i, u_i)_{i \in I}$ is uniformly payoff secure and its aggregator function A_G^0 is upper semicontinuous, then the mixed extension $\Gamma(G)$ is diagonally transfer continuous, and, therefore, G possesses a mixed strategy Nash equilibrium.

The notion of a uniformly payoff secure game can be extended to diagonally transfer continuous games.

Definition 7 A game $G = (X_i, u_i)_{i \in I}$ is uniformly diagonally secure if, for every $d \in X$ and every $\varepsilon > 0$, there is $\bar{d} \in X$ such that for every $x \in X$, $F_G(\bar{d}, w) > F_G(d, x) - \varepsilon$ for all w in some open neighborhood $\mathcal{N}_X(x)$ of x in X .

An upper semicontinuous-sum game G is uniformly diagonally secure if it is uniformly payoff secure.

Lemma 2 If a game $G = (X_i, u_i)_{i \in I}$ is uniformly payoff secure and the aggregator function $A_G^0 : X \rightarrow \mathbb{R}$ is upper semicontinuous, then G is uniformly diagonally secure.

Proof. Fix $d \in X$ and $\varepsilon > 0$. By the uniform payoff security of G , for each $i \in I$, there is a deviation strategy $\bar{d}_i \in X_i$ such that for every $x_{-i} \in X_{-i}$, $u_i(\bar{d}_i, w_{-i}) \geq u_i(d_i, x_{-i}) - \frac{\varepsilon}{2n}$ for all w_{-i} in some open neighborhood $\mathcal{N}_{X_{-i}}(x_{-i})$ of x_{-i} in X_{-i} . Denote $\mathcal{N}_X^1(x) = \bigcap_{i \in I} \{X_i \times \mathcal{N}_{X_{-i}}(x_{-i})\}$ for $x \in X$. Then $A_G(\bar{d}, w) \geq A_G(d, x) - \frac{\varepsilon}{2}$ for all $w \in \mathcal{N}_X^1(x)$. Since the function $-A_G^0$ is lower semicontinuous on X , there exists a neighborhood $\mathcal{N}_X^2(x)$ such that $-A_G^0(w) > -A_G^0(x) - \frac{\varepsilon}{2}$ for all $w \in \mathcal{N}_X^2(x)$. Then for every $x \in X$, $F_G(\bar{d}, w) > F_G(d, x) - \varepsilon$ for all $w \in \mathcal{N}_X^1(x) \cap \mathcal{N}_X^2(x)$. ■

Theorem 3 *If a game $G = (X_i, u_i)_{i \in I}$ is uniformly diagonally secure, then its mixed extension $\Gamma(G)$ is diagonally transfer continuous, and, therefore, G possesses a mixed strategy Nash equilibrium.*

The proof of Theorem 3 follows the lines of the proof of Theorem 1 of Monteiro and Page (2007) and is given in the Appendix.

The concept of uniform diagonal security might be of help in studying equilibrium existence in games whose aggregator function A_G^0 is not upper semicontinuous.

Example 1 Consider a slight modification of the rent-seeking game due to Tullock (1980). Two players simultaneously bid for a political favor commonly known worth V dollars. Their bids, denoted by x_1 and x_2 , influence the probability of receiving the favor. Player i 's strategy set is the segment $[0, V]$. Let $\pi_i(x_1, x_2)$ denote the probability that player i wins. The function π_i , called player i 's contest success function, is specified as follows:

$$\pi_i(x_i, x_{-i}) = \begin{cases} \frac{1}{4} & \text{if } x_1 = x_2 = 0, \\ \frac{1}{2} & \text{if } x_i > x_{-i} = 0, \\ \frac{x_i^r}{x_i^r + x_{-i}^r} & \text{otherwise,} \end{cases}$$

where $r > 0$. Player i 's payoff function u_i is

$$u_i(x_i, x_{-i}) = \pi_i(x_i, x_{-i})V - x_i.$$

The only difference of the model from the Tullock rent-seeking game is the assumption that if the lowest bid submitted is equal to zero (or, in other words, at least one player exerts no effort at all), the favor may be granted to a third party with probability one-half. Consequently, the aggregator function A_G^0 is not upper semicontinuous.

Let, for specificity, $V = 2$ and $r = 3$. In this case, the game has no pure strategy Nash equilibria (see, for a related discussion, Baye, Kovenock, and de Vries, 1994). For instance, one can check that the only candidate point for being an interior solution is $(1.5, 1.5)$, a strategy profile where both players get negative expected payoffs. However, each of them can avoid getting a negative payoff by bidding zero.

To verify that the game is not better-reply secure, consider the sequence $\{x^k\}$ with $x^k = (\frac{1}{k}, \frac{1}{k})$ for $k = 1, 2, \dots$. Then the corresponding sequence of payoff vectors $\{(u_1(x^k), u_2(x^k))\}$ converges to $(1, 1)$. It is clear that no player can secure a payoff strictly above 1 at $(0, 0)$.

On the other hand, the game possesses mixed strategy equilibria since it is uniformly diagonally secure. To verify this, for $d \in X$ and $\varepsilon \in (0, 1)$, define $\bar{d} = (\bar{d}_1, \bar{d}_2)$ as follows:

$$\bar{d}_i = \begin{cases} d_i & \text{if } d_i > 0, \\ \frac{\varepsilon}{4} & \text{if } d_i = 0, \end{cases} \quad \text{for } i = 1, 2.$$

It is a little tedious but not difficult to show that, for every $x \in X$, $F_G(\bar{d}, w) > F_G(d, x) - \varepsilon$ for all w in some open neighborhood $\mathcal{N}_X(x)$.

4 Weak Uniform Payoff Security

The Sion-Wolfe (1957) game is a better-reply secure game that has no Nash equilibria. Its mixed extension is upper semicontinuous-sum, but is not payoff secure (see Carmona, 2005, Example 3). Further on, we study equilibrium existence in games whose mixed extension is payoff secure. Example 2 shows

that there are better-reply secure games with a payoff secure mixed extension which have no Nash equilibria.

The following definition extends Simon's (1987, p. 577) concept of weak domination on average to our setting.

Definition 8 *In $G = (X_i, u_i)_{i \in I}$, player i 's payoff function u_i is weakly uniformly transfer lower semicontinuous in the other players' strategies at $x_i \in X_i$ if, for every $\mu_{-i} \in \Delta(X_{-i})$ and every $\varepsilon > 0$, there are a strategy $\sigma_i(x_i, \mu_{-i}, \varepsilon) \in \Delta(X_i)$ and a Borel set $Q_{-i} \subset X_{-i}$ with $\mu_{-i}(Q_{-i}) > 1 - \varepsilon$ such that for every $x_{-i} \in Q_{-i}$, $U_i(\sigma_i(x_i, \mu_{-i}, \varepsilon), w_{-i}) \geq u_i(x_i, x_{-i}) - \varepsilon$ for all w_{-i} in some open neighborhood $\mathcal{N}_{X_{-i}}(x_{-i})$ of x_{-i} in X_{-i} . The game G is weakly uniformly payoff secure if each u_i is weakly uniformly transfer lower semicontinuous in the other players' strategies at every $x_i \in X_i$.*

There is some notational abuse in denoting $\int_{X_i} u_i(x_i, w_{-i}) d\sigma_i(x_i, \mu_{-i}, \varepsilon)$ by $U_i(\sigma_i(x_i, \mu_{-i}, \varepsilon), w_{-i})$, but no ambiguity ensues. The proof of the next statement follows the lines of the proof of Proposition 3 of Simon (1987) and is relegated to the Appendix.

Lemma 3 *If a game $G = (X_i, u_i)_{i \in I}$ is weakly uniformly payoff secure, then its mixed extension $\Gamma(G)$ is payoff secure.*

The game studied in Example 2 is not only better-reply secure but also has a payoff secure mixed extension. However it does not possess mixed strategy Nash equilibria.

Example 2 Let $I = \{1, 2\}$, $X = [0, 1] \times [0, 1]$, and the payoff functions

are defined by

$$u_1(x_1, x_2) = \begin{cases} 0 & \text{if } x_2 = x_1 + \frac{1}{2}, x_2 = x_1, \text{ or } (x_1, x_2) = (0, 1), \\ x_2 - x_1 - 1 & \text{if } 0 \leq x_1 < x_2 < x_1 + \frac{1}{2}, \\ 1 & \text{otherwise,} \end{cases}$$

$$u_2(x_2, x_1) = \begin{cases} 0 & \text{if } x_2 = x_1 + \frac{1}{2}, x_2 = x_1, \text{ or } (x_1, x_2) \in [0, \frac{1}{2}) \times \{1\}, \\ 1 & \text{if } 0 \leq x_1 < x_2 < x_1 + \frac{1}{2}, \\ -1 & \text{otherwise.} \end{cases}$$

Let us show that the game has no mixed strategy Nash equilibria. It is clear that there is no Nash equilibrium $(\hat{\mu}_1, \hat{\mu}_2)$ with $\hat{\mu}_2(\{1\}) = 1$.

Assume, by way of contradiction, that $(\hat{\mu}_1, \hat{\mu}_2)$ is a mixed strategy Nash equilibrium of the game.

If $\hat{\mu}_2(\{1\}) = 0$, then the set $D = \{(x_1, x_2) \in X : \min\{0, x_2 - \frac{1}{2}\} \leq x_1 \leq x_2\}$ must be a null set with respect to the product measure $\hat{\mu} = \hat{\mu}_1 \times \hat{\mu}_2$, otherwise, by choosing strategy 1 with probability 1, player 1 can increase her payoff. Then, in the equilibrium, the payoff to player 2 is equal to -1 . However, player 2 can guarantee herself at least 0 by choosing strategy 1 with probability 1, a contradiction. Therefore, $\hat{\mu}_2(\{1\}) > 0$. Then it must also be the case that $\hat{\mu}_1((\frac{1}{2}, 1)) = 0$.

The set $D_1 = \{(x_1, x_2) \in X : 0 \leq x_2 \leq x_1, 0 \leq x_1 < \frac{1}{2}\}$ must be a null set with respect to the product measure $\hat{\mu}$, otherwise player 2 can increase her payoff by shifting some weight from $[0, \frac{1}{2})$ closer to $\frac{1}{2}$. The set $D_2 = \{(x_1, x_2) \in X : x_1 < x_2 \leq x_1 + \frac{1}{2}, 0 < x_1 < \frac{1}{2}\}$ is also a null set, otherwise player 1 could benefit from shifting some weight from $(0, \frac{1}{2})$ closer to 0 (recall that, on D_2 , $u_1(\cdot, x_2)$ is a strictly decreasing function for any $x_2 \in (0, 1)$). Furthermore, the set $D_3 = \{(x_1, x_2) \in X : x_1 + \frac{1}{2} < x_2 < 1, x_1 \neq 0\}$ is a null set, otherwise player 2 can increase her payoff by shifting some weight from $(\frac{1}{2}, 1)$ closer to $\frac{1}{2}$.

If $\hat{\mu}_1((0, \frac{1}{2})) > 0$, then the fact that $D_1 \cup D_2 \cup D_3$ is a null set implies that $\hat{\mu}_2([0, 1)) = 0$, which is impossible since there is no Nash equilibrium with

$\hat{\mu}_2(\{1\}) = 1$. Thus, $\hat{\mu}_1(\{0\} \cup \{\frac{1}{2}\} \cup \{1\}) = 1$.

If $\hat{\mu}_2([0, \frac{1}{2}]) > 0$, then $\hat{\mu}_1(\{0\}) = 0$, otherwise player 1 can increase her payoff by shifting the mass $\hat{\mu}_1(\{0\})$ to strategy 1 (recall that $u(0, 1) = 0$). Further, player 2 can increase her payoff by shifting the mass $\hat{\mu}_2([0, \frac{1}{2}])$ to strategy 1 if $\hat{\mu}_1(\{1\}) > 0$, and to $\frac{2}{3}$ if $\hat{\mu}_1(\{1\}) = 0$, a contradiction.

If $\hat{\mu}_2((\frac{1}{2}, 1)) > 0$, then $\hat{\mu}_1(\{\frac{1}{2}\}) = 0$ (since $\hat{\mu}_2((\frac{1}{2}, 1]) = 1$), and again player 2 can get a nonnegative payoff by playing 1 with probability 1. Thus, $\hat{\mu}_2(\{1\}) = 1$, a contradiction.

We will now show, for completeness, that u_1 is weakly uniformly transfer lower semicontinuous in player 2's strategies at every $x_1 \in X_1$. Denote by δ_{x_1} the Dirac measure concentrated at x_1 . One can see that the payoff function u_1 is uniformly payoff secure at any $x_1 \in \{0\} \cup [\frac{1}{2}, 1]$, with $\sigma_1(x_1, \mu_2, \varepsilon) = \delta_{x_1}$ for every $\mu_2 \in \Delta(X_2)$, and every $\varepsilon > 0$. Fix some $x_1 \in (0, \frac{1}{2})$, $\mu_2 \in \Delta(X_2)$, and $\varepsilon > 0$. It is not difficult to see that there is $\varepsilon' \in (0, \varepsilon)$ such that both $x_1 - \varepsilon'$ and $x_1 + \varepsilon'$ lie in the interval $(0, \frac{1}{2})$, $\mu_2(Q_2) > 1 - \varepsilon$ where $Q_2 = X_2 \setminus ([x_1 - \varepsilon', x_1] \cup (x_1, x_1 + \varepsilon'] \cup [x_1 + \frac{1}{2} - \varepsilon', x_1 + \frac{1}{2}] \cup (x_1 + \frac{1}{2}, x_1 + \frac{1}{2} + \varepsilon'])$, and $U_1(\frac{1}{2}\delta_{x_1 - \varepsilon'} + \frac{1}{2}\delta_{x_1 + \varepsilon'}, w_2) \geq u_1(x_1, x_2) - \varepsilon$ for every $x_2 \in Q_2$ and all w_2 in some open neighborhood $\mathcal{N}_{X_2}(x_2)$ of x_2 in X_2 . A similar reasoning can be used to show that u_2 is weakly uniformly transfer lower semicontinuous in player 1's strategies at every $x_2 \in X_2$.

5 Equilibria of Reciprocally Upper Semicontinuous Games with Payoff Secure Mixed Extensions

Verifying whether a game is transfer reciprocally upper semicontinuous is often considerably easier than doing that for its mixed extension. In this section, we study several classes of better-reply secure games with payoff secure mixed extensions that also have transfer reciprocally upper semicontinuous

mixed extensions.

Some reciprocally upper semicontinuous games can be transformed into upper semicontinuous-sum games with the aid of positive affine transformations. If this is the case, the game has a reciprocally upper semicontinuous mixed extension. Example 3 demonstrates that a conventional probabilistic spatial voting model possesses this property.

Then we study equilibrium existence in reciprocally upper semicontinuous games of two players on the unit square that have a payoff secure mixed extension. Additional conditions are made on the set of discontinuities of the payoff functions. In particular, it is assumed that the discontinuities lie on one or two strictly monotone curves.

We now introduce some notation. Recall that a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a positive affine transformation if it can be written in the form: $\phi(x) = ax + b$ where $a > 0$ and b is any real number. Given a game $G = (X_i, u_i)_{i \in I}$ and an n -tuple of positive affine transformations (ϕ_1, \dots, ϕ_n) , we denote by $\Phi(G)$ the game $(X_i, \phi_i(u_i))_{i \in I}$. The following three facts are straightforward.

(i) Let $G = (X_i, u_i)_{i \in I}$, and let (ϕ_1, \dots, ϕ_n) be an n -tuple of positive affine transformations. Then $E_{\Gamma(G)} = E_{\Gamma(\Phi(G))}$.

(ii) Let (ϕ_1, \dots, ϕ_n) be an n -tuple of positive affine transformations. Then $G = (X_i, u_i)_{i \in N}$ is weakly uniformly payoff secure iff the game $\Phi(G)$ is weakly uniformly payoff secure.

(iii) Let (ϕ_1, \dots, ϕ_n) be an n -tuple of positive affine transformations. Then $G = (X_i, u_i)_{i \in N}$ is reciprocally upper semicontinuous iff the game $\Phi(G)$ is reciprocally upper semicontinuous.

Theorem 4 *If $G = (X_i, u_i)_{i \in I}$ is weakly uniformly payoff secure, and there is an n -tuple of positive affine transformations (ϕ_1, \dots, ϕ_n) such that the game $\Phi(G)$ is upper semicontinuous-sum, then G possesses a mixed strategy Nash equilibrium.*

Proof. Since G is weakly uniformly payoff secure, the game $\Phi(G)$ is also

weakly uniformly payoff secure by fact (ii). Thus, $\Gamma(\Phi(G))$ is payoff secure by Lemma 3.

The fact that $\Phi(G)$ is upper semicontinuous-sum implies that $\Gamma(\Phi(G))$ is upper semicontinuous-sum. Since $\Gamma(\Phi(G))$ is the same game as $\Phi(\Gamma(G))$, the mixed extension $\Gamma(G)$ is reciprocally upper semicontinuous by fact (iii). By Theorem 1, $\Gamma(G)$ possesses a pure strategy Nash equilibrium. ■

Example 3 Consider the following spatial voting model (see Ball, 1999, Example 1). Two candidates are competing in an election for public office. The electorate is distributed uniformly along the ideological spectrum $[0, 1]$. During the electoral campaign, each candidate i announces, simultaneously with the other candidate, a platform, denoted by x_i . The probability $P_i(x_i, x_{-i})$ that candidate i wins the election is defined as follows:

$$P_i(x_i, x_{-i}) = \begin{cases} \frac{x_i + x_{-i}}{2} & \text{for } 0 \leq x_i < x_{-i} \leq 1, \\ \frac{1}{2} & \text{for } 0 \leq x_i = x_{-i} \leq 1, \\ 1 - \frac{x_i + x_{-i}}{2} & \text{for } 0 \leq x_{-i} < x_i \leq 1. \end{cases}$$

Candidates 1 and 2's policy preferences on $[0, 1]$ are represented by $h_1(z) = -\frac{1}{2}(z - 1)^2$ and $h_2(z) = -\frac{1}{2}z^2$. The candidates are assumed to be office-motivated. Let the candidates' office motivation parameters be $k_1 = .05$ and $k_2 = 3$, respectively. Candidate i 's payoff function is

$$u_i(x_i, x_{-i}) = P_i(x_i, x_{-i})(h_i(x_i) + k_i) + (1 - P_i(x_i, x_{-i}))h_i(x_{-i}).$$

The game is not upper semicontinuous-sum and has no pure strategy Nash equilibria.

Now we consider the game $\Phi(G) = (X_i, \phi_i(u_i))_{i \in \{1, 2\}}$ where ϕ_1 and ϕ_2 are defined by $\phi_1(t) = \frac{k_2}{k_1}t$ and $\phi_2(t) = t$ for $t \in \mathbb{R}$, respectively. It is not difficult to see that $A_{\Phi(G)}^0$ is a continuous function on X . Since G is uniformly payoff secure, G possesses a mixed strategy Nash equilibrium by Theorem 4.

It is useful to notice that the voting game is an example of a game with a better-reply secure mixed extension that is not upper semicontinuous-sum.

In the next three theorems, we describe some representative reciprocally upper semicontinuous games of two persons on the unit square with payoff secure mixed extensions that whose mixed extensions are also better-reply secure. The additional assumptions will be made on the set of discontinuities of the payoff functions. To start with, we consider the case when all discontinuities of the payoff functions lie on a single strictly monotonic curve. Denote $S^1 = \{(x_1, x_2) \in [0, 1] \times [0, 1] : x_2 > x_1\}$, $S^2 = \{(x_1, x_2) \in [0, 1] \times [0, 1] : x_1 > x_2\}$, and $S = \{(x_1, x_2) : [0, 1] \times [0, 1] : x_1 = x_2\}$.

Theorem 5 *Let $G = (X_i, u_i)_{i \in \{1, 2\}}$ be a two-player game on the unit square $X_1 \times X_2 = [0, 1] \times [0, 1]$. Suppose that*

- (i) *G is uniformly payoff secure and reciprocally upper semicontinuous;*
- (ii) *there are continuous functions $f_i^j : \text{cl}S^j \rightarrow \mathbb{R}$, $i = 1, 2$, $j = 1, 2$, such that $u_i(x) = f_i^j(x)$ for all $x \in S^j$ and all $i, j \in \{1, 2\}$;*
- (iii) *$u_1(0, 0) \leq f_1^2(0, 0)$ and $u_1(1, 1) \leq f_1^1(1, 1)$;*
- (iv) *the restriction of each u_i to S , $u_{i|S}$, is a continuous function from S to \mathbb{R} .*

Then G has a better-reply secure mixed extension, and, therefore, it has a mixed strategy Nash equilibrium.

Condition (ii) is stronger than the assumption that all points of discontinuity of the payoff functions lie on S . In particular, (ii) implies that $\lim_{k \rightarrow \infty} u_i(x^k) = f_i^j(x)$ for every sequence of points $\{x^k\}$, $x^k \in S^j$, that converges to $x \in S$; that is, the limit value does not depend on the choice of a sequence in S^j . Condition (iii) may be replaced, if needed, with a number of other conditions, such as: (iii') for every $\varepsilon > 0$ and each $z \in \{0, 1\}$, there exists a deviation strategy $d_1(z, \varepsilon)$ satisfying the uniform payoff security condition for player 1's strategy z that is different from z ; or (iii'') $u_2(0, 0) \leq f_2^1(0, 0)$ and $u_2(1, 1) \leq f_2^2(1, 1)$.

Theorem 5 can be extended to a class of games where the discontinuities of the payoff functions might lie on two strictly monotone disjoint

curves. In particular, Theorem 6 covers the games where one of the curves is strictly increasing and the other is strictly decreasing.⁸ We will show it for a representative game from this class. For every $h : \mathbb{R}^1 \rightarrow \mathbb{R}^1$, denote $X_h^1 = \{(x_1, x_2) : h(x_1) < x_2\}$, $X_h^2 = \{(x_1, x_2) : h(x_1) > x_2\}$, and $X_h = \{(x_1, x_2) : h(x_1) = x_2\}$.

Theorem 6 *Let $G = (X_i, u_i)_{i \in \{1,2\}}$ be a two-player game on the unit square $X_1 \times X_2 = [0, 1] \times [0, 1]$, and let $h_1 : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ be defined by $h_1(x_1) = \frac{1}{4}x_1 + \frac{3}{4}$ and $h_2 : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ by $h_2(x_1) = -\frac{1}{4}x_1 + \frac{1}{4}$. Suppose that*

- (i) *G is uniformly payoff secure and reciprocally upper semicontinuous;*
- (ii) *for each $i \in \{1, 2\}$, there are continuous functions $f_i^1 : \text{cl}X_{h_1}^1 \rightarrow \mathbb{R}$, $f_i^2 : \text{cl}X_{h_1}^2 \cap \text{cl}X_{h_2}^1 \rightarrow \mathbb{R}$, and $f_i^3 : \text{cl}X_{h_2}^2 \rightarrow \mathbb{R}$ such that $u_i(x) = f_i^1(x)$ for every $x \in X_{h_1}^1$, $u_i(x) = f_i^2(x)$ for every $x \in \text{cl}X_{h_1}^2 \cap \text{cl}X_{h_2}^1$, and $u_i(x) = f_i^3(x)$ for every $x \in X_{h_2}^2$;*
- (iii) *$u_1(1, 1) \leq f_1^1(1, 1)$ and $u_1(1, 0) \leq f_1^3(1, 0)$;*
- (iv) *for each $i \in \{1, 2\}$ and each $l \in \{1, 2\}$, the restriction of u_i to X_{h_l} , $u_i|_{X_{h_l}}$, is a continuous function from X_{h_l} to \mathbb{R} .*

Then G has a better-reply secure mixed extension, and, therefore, it has a mixed strategy Nash equilibrium.

Condition (iii) of Theorem 6 may be replaced with a number of other geometric conditions, if need appears.

Example 4 Consider the following deterministic spatial voting model. Two candidates are competing in an election for public office. The electorate is distributed uniformly along the ideological spectrum $[0, 1]$. The candidate with the most votes wins. Each voter casts his vote for the candidate that is closest to her ideological position. During the electoral campaign, each candidate i announces, simultaneously with the other candidate, a platform $x_i \in [0, 1]$ to which he commits if elected. The probability that candidate i

⁸Equilibrium existence conditions similar to those presented in Theorem 6 can also be provided for Sion-Wolfe-type games.

wins the election is

$$P_i(x_i, x_{-i}) = \begin{cases} 1, & \text{if } (x_i + x_{-i} - 1)(x_{-i} - x_i) > 0, \\ \frac{1}{2} & \text{if } \frac{x_i + x_{-i}}{2} = \frac{1}{2}, \\ 0, & \text{if } (x_i + x_{-i} - 1)(x_{-i} - x_i) < 0. \end{cases}$$

The inequality $(x_i + x_{-i} - 1)(x_{-i} - x_i) > 0$ represents the fact that candidate i gets most of the votes. For example, if $x_{-i} > x_i$ and $\frac{x_i + x_{-i}}{2} > \frac{1}{2}$, then $(x_i + x_{-i} - 1)(x_{-i} - x_i) > 0$. The candidates' loss functions on $[0, 1]$ are represented by $h_1(z) = -\frac{1}{2}(z - 1)^2$ and $h_2(z) = -\frac{1}{2}z^2$, and the candidates are assumed to be office-motivated. Let the office motivation parameters be $k_1 = .05$ and $k_2 = 3$, respectively. Candidate i 's payoff function is

$$u_i(x_i, x_{-i}) = h_i(x_i) + P_i(x_i, x_{-i})k_i.$$

Thus, each candidate cares only about winning, and taking an ideological position different from her ideal point is costly. It is not difficult to see that the game has no pure strategy Nash equilibria.

Despite the fact that all discontinuities of the payoff functions lie on the intersecting diagonals of the unit square, it is possible to use Theorem 6 to show that the game has mixed strategy equilibria. First notice that every pure strategy $z \in [0, \frac{3}{5})$ of player 1 is strictly dominated by strategy 1. As a result, to show that the game under study has mixed strategy equilibria, it is enough to investigate whether its restriction G^r to $[\frac{3}{5}, 1] \times [0, 1]$ has mixed strategy equilibria, where the game $G^r = (X_i^r, u_i^r)_{i \in \{1, 2\}}$ is defined as follows: $X_1^r = [\frac{3}{5}, 1]$, $X_2^r = X_2 = [0, 1]$, $u_i^r(x) = u_i(x)$ for every $x \in X_1^r \times X_2^r$ and each i .

One can see that G^r is uniformly payoff secure. For example, given $\varepsilon > 0$, if $x_1 = \frac{3}{5}$, then for every $x_2 \in [0, 1]$ there exists an open neighborhood $\mathcal{N}_{X_2}(x_2)$ such that $u_1(1, w_2) > u_1(\frac{3}{5}, x_2) - \varepsilon$ for all $w_2 \in \mathcal{N}_{X_2}(x_2)$. If $x_1 \in (\frac{3}{5}, 1]$, then there is $\delta \in (0, x_1 - \frac{3}{5})$ such that for every $x_2 \in [0, 1]$, $u_1(x_1 - \delta, w_2) >$

$u_1(x_1, x_2) - \varepsilon$ for all w_2 in some neighborhood $\mathcal{N}_{X_2}(x_2)$ of x_2 . Clearly, the game is reciprocally upper semicontinuous. The existence of a mixed strategy Nash equilibrium in G^r follows from Theorem 6.

6 Conclusions

We use aggregator functions in conjunction with the Ky Fan minimax inequality to study the existence of a mixed strategy Nash equilibrium in diagonally transfer continuous games (Baye, Tian, and Zhou, 1993). Similar to the approach based on the concept of better-reply security (Reny, 1999), the aggregation-based approach is applicable to the upper semicontinuous-sum games whose mixed extension is payoff secure. However, if the sum of the payoff functions of a game is not upper semicontinuous, showing that its mixed extension is either better-reply secure or diagonally transfer continuous often constitutes an intractable problem. To alleviate it, the concept of uniform payoff security (Monteiro and Page, 2007) is extended to diagonally transfer continuous games by introducing uniform diagonal security. We show that every uniformly diagonally secure game possesses a mixed strategy Nash equilibrium, and, with the aid of an example, that uniformly diagonally secure games need not be better-reply secure.

After introducing a generalization of uniform payoff security, called weak uniform payoff security, we provide an example of a better-reply secure game with a payoff secure mixed extension that has no mixed strategy Nash equilibria. Then we study the existence of a mixed strategy Nash equilibrium in games that are reciprocally upper semicontinuous and uniformly payoff secure. We propose two sets of easily verifiable, geometric conditions for a two-person game to possess a better-reply secure mixed extension.

Appendix

Proof of Lemma 1

We shall first show that $A_G(d, x)$ is transfer lower semicontinuous in x . Let $(d, x) \in X \times X$ and $\lambda \in \mathbb{R}$ be such that $A_G(d, x) > \lambda$. Then there are $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\lambda = \lambda_1 + \dots + \lambda_n$ and $u_i(d_i, x_{-i}) > \lambda_i$ for all $i \in I$. Since each u_i is transfer lower semicontinuous in x_{-i} , there exist $\bar{d}_i \in X_i$ and an open neighborhood $\mathcal{N}_{X_{-i}}(x_{-i})$ of x_{-i} in X_{-i} such that $u_i(\bar{d}_i, z_{-i}) > \lambda_i$ for all $z_{-i} \in \mathcal{N}_{X_{-i}}(x_{-i})$. Consequently, $A_G(\bar{d}, z) > \lambda$ for every $z \in \bigcap_{i \in I} \{X_i \times \mathcal{N}_{X_{-i}}(x_{-i})\}$.

Since A_G^0 is upper semicontinuous on X , the transfer lower semicontinuity of A_G in x implies the transfer lower semicontinuity of F_G in x . Therefore F_G is 0-transfer lower semicontinuous in x .

The Ky Fan Minimax Inequality

In Theorem 2, the Ky Fan minimax inequality is used in the following, slightly generalized form. (see, e.g., Ding and Park, 2002; Lan and Wu, 2002, for more general results).

Lemma 4 *Let X be a compact convex set in a Hausdorff topological vector space, and let $f : X \times X \rightarrow \mathbb{R}$ satisfy:*

- (i) $f(x, x) \leq 0$ for each $x \in X$;
- (i) $f(\cdot, y)$ is quasiconcave for each $y \in X$,
- (ii) f is 0-transfer lower semicontinuous in y .

Then there exists $\bar{y} \in X$ such that $f(x, \bar{y}) \leq 0$ for all $x \in X$.

Lemma 4 can be shown in a number of ways. Its conventional proofs are based either on the KKM lemma or on Browder's fixed point theorem, which are two equivalent results (see, for an in-depth discussion, Yannelis, 1991). Let us give an outline of the proof using Browder's fixed point theorem. It

proceeds by assuming, to the contrary, that, for each $y \in X$, there exists $x \in X$ such that $f(x, y) > 0$. Then the correspondence $M : X \rightrightarrows X$ defined by $M(y) = \{x \in X : f(x, y) > 0\}$ has nonempty values. The quasiconcavity of f in x implies that M has convex values. Since f is 0-transfer lower semicontinuous in y , M has a multivalued selection with open lower sections (see, e.g., Prokopovych, 2011), denoted by $M_0 : X \rightrightarrows X$. Then, by Lemma 5.1 of Yannelis and Prabhakar (1983), the convex-valued correspondence $\overline{M}_0 : X \rightrightarrows X$ defined by $\overline{M}_0(x) = \text{co}M_0(x)$ also has open lower sections. Therefore, by Browder's fixed point theorem, the selection has a fixed point, which contradicts (i).

Proof of Theorem 3

Suppose $\mu = (\mu_1, \dots, \mu_n) \in \Delta(X) \setminus E_{\Gamma(G)}$. Then there exists $\sigma \in \Delta(X)$ such that $F_{\Gamma(G)}(\sigma, \mu) > 0$. Since σ is a vector of probability measures, there exists $d = (d_1, \dots, d_n) \in X$ such that $F_{\Gamma(G)}(\delta_d, \mu) > 0$, where $\delta_d = (\delta_{d_1}, \dots, \delta_{d_n})$ is the vector of Dirac measures concentrated at d_1, \dots, d_n , respectively. With some abuse of notation, we will write $F_{\Gamma(G)}(d, \mu)$ in place of $F_{\Gamma(G)}(\delta_d, \mu)$. Put $\varepsilon^* = F_{\Gamma(G)}(d, \mu)$ and denote $\liminf_{w \rightarrow x} F_G(d, w)$ by $\underline{F}_G(d, x)$. Since G is uniformly diagonally secure, there is $\bar{d} \in X$ such that $\underline{F}_G(\bar{d}, x) > F_G(d, x) - \frac{\varepsilon^*}{2}$ for all $x \in X$. Therefore $\underline{F}_{\Gamma(G)}(\bar{d}, \mu) > F_{\Gamma(G)}(d, \mu) - \frac{\varepsilon^*}{2}$, where $\underline{F}_{\Gamma(G)}(\bar{d}, \mu) = \int_X \underline{F}_G(\bar{d}, x) d\mu$.

The lower semicontinuity of \underline{F}_G in x implies that $\underline{F}_{\Gamma(G)}(\bar{d}, \cdot) : \Delta(X) \rightarrow \mathbb{R}$ is lower semicontinuous (see, e.g., Aliprantis and Border, 2006, Theorem 15.5). Consequently, $\underline{F}_{\Gamma(G)}(\bar{d}, \mu') > F_{\Gamma(G)}(d, \mu) - \frac{\varepsilon^*}{2} > 0$ for all μ' in some open neighborhood $\mathcal{N}_{\Delta(X)}(\mu)$ of μ . Since $F_G(\bar{d}, x) \geq \underline{F}_G(\bar{d}, x)$ for all $x \in X$, we conclude that $F_{\Gamma(G)}(\bar{d}, \mu') \geq \underline{F}_{\Gamma(G)}(\bar{d}, \mu') > 0$ for all $\mu' \in \mathcal{N}_{\Delta(X)}(\mu)$, which means that $\Gamma(G)$ is diagonally transfer continuous.

Proof of Lemma 3

Fix some $\mu = (\mu_1, \dots, \mu_n) \in \Delta(X)$, $\varepsilon > 0$, and $i \in I$. We have to show that, for each $i \in I$, there are a strategy $\sigma_i \in \Delta(X_i)$ and a neighborhood $\mathcal{N}_{\Delta(X_{-i})}(\mu_{-i})$ such that $U_i(\sigma_i, \mu'_{-i}) \geq u_i(\mu) - \varepsilon$ for all $\mu'_{-i} \in \mathcal{N}_{\Delta(X_{-i})}(\mu_{-i})$.

Fix some $i \in I$. There is $x_i \in X_i$ such that $U_i(x_i, \mu_{-i}) > u_i(\mu) - \frac{\varepsilon}{4}$. Since G is a bounded game, there exists $B > 1$ such that $|u_i(x)| < B$ for every $x \in X$. Pick $\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}) \in \Delta(X_i)$ and a Borel set $Q_{-i} \subset X_{-i}$ with $\mu_{-i}(Q_{-i}) > 1 - \frac{\varepsilon}{4B}$ such that $\liminf_{w_{-i} \rightarrow x_{-i}} U_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), w_{-i}) \geq u_i(x_i, x_{-i}) - \frac{\varepsilon}{4B}$ for every $x_{-i} \in Q_{-i}$. For the brevity of notation, denote $\liminf_{w_{-i} \rightarrow x_{-i}} U_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), w_{-i})$ by $\underline{U}_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), x_{-i})$. Then

$$\begin{aligned} \int \underline{U}_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), x_{-i}) d\mu_{-i} &\geq \int_{Q_{-i}} \underline{U}_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), x_{-i}) d\mu_{-i} - \frac{\varepsilon}{4} \\ &\geq \int_{Q_{-i}} u_i(x_i, x_{-i}) d\mu_{-i} - \frac{\varepsilon}{2} \geq U_i(x_i, \mu_{-i}) - \frac{3\varepsilon}{4} > U_i(\mu) - \varepsilon \end{aligned}$$

The lower semicontinuity of $\underline{U}_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), \cdot)$ implies that $\underline{U}_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), \cdot) : \Delta(X_{-i}) \rightarrow \mathbb{R}$ defined by $\underline{U}_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), \mu_{-i}) = \int \underline{U}_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), x_{-i}) d\mu_{-i}$ is lower semicontinuous in the second argument (see, e.g., Aliprantis and Border, 2006, Theorem 15.5). Consequently, $\underline{U}_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), \mu'_{-i}) > U_i(\mu) - \varepsilon$ for all μ'_{-i} in some open neighborhood $\mathcal{N}_{\Delta(X_{-i})}(\mu_{-i})$ of μ_{-i} . On the other hand, since $U_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), \mu'_{-i}) \geq \int \underline{U}_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), x_{-i}) d\mu'_{-i}$ for every $\mu'_{-i} \in \Delta(X_{-i})$, we have that $U_i(\sigma_i(x_i, \mu_{-i}, \frac{\varepsilon}{4B}), \mu'_{-i}) > U_i(\mu) - \varepsilon$ for every $\mu'_{-i} \in \mathcal{N}_{\Delta(X_{-i})}(\mu_{-i})$, which completes the proof.

An Auxiliary Lemma

Lemma 5 *Let $X = [0, 1] \times [0, 1]$, and $\mu_i \in \Delta([0, 1])$, $i = 1, 2$, and let D be a Borel subset of $[0, 1]$. Then the set $\bar{D} = \{(z, z) \in [0, 1] \times [0, 1] : z \in D\}$ is a null set with respect to the product measure $\mu = \mu_1 \times \mu_2$ iff $\mu(x) = 0$ for every $x \in \bar{D}$.*

Proof. Assume that $\mu(x) = 0$ for every $x \in \overline{D}$. We have to show that $\mu(\overline{D}) = 0$.

First consider the case where, for some $i \in \{1, 2\}$, $\mu_i(x) = 0$ for all $x \in D$. Fix some arbitrary $\varepsilon > 0$. Since μ_i has no mass points on D , there exists a finite disjoint collection $\{D_k\}_{k=1}^m$ of subsets of D such that $D = \cup_{k=1}^m D_k$ and $\mu_i(D_k) \leq \varepsilon$ for all k . Then $\mu(\overline{D}) \leq \sum_k \mu(D_k \times D_k) = \sum_k \mu_i(D_k) \mu_{-i}(D_k) \leq \varepsilon \mu_{-i}(D) \leq \varepsilon$, which implies that $\mu(\overline{D}) = 0$.

Let both μ_1 and μ_2 have mass points on D . Each μ_i , as a finite measure, can have at most a countable number of mass points. Denote by C_i the set of mass points of μ_i on D . Since $\mu(x) = 0$ for every $x \in \overline{D}$, $C_1 \cap C_2 = \emptyset$.

Fix some arbitrary $\varepsilon > 0$. Since $C_1 \cap C_2 = \emptyset$, it is possible to associate with every $x \in C_1$ a Borel set $D(x) \subset [0, 1]$ containing x such that $\mu_1(D(x)) \leq 2\mu_1(x)$ and $\mu_2(D(x)) \leq \frac{\varepsilon}{2}$. Let $D_1 = D \setminus C_1$. The set D_1 is Borel, and μ_1 is nonatomic on it. Denote by $\overline{C}_1(\overline{D}_1)$ the set $\{(z, z) \in [0, 1] \times [0, 1] : z \in C_1(z \in D_1)\}$. It is clear that \overline{C}_1 is at most countable and $\mu(\overline{D}_1) = 0$. Then, since $\overline{D} = \overline{C}_1 \cup \overline{D}_1$ and $\overline{C}_1 \cap \overline{D}_1 = \emptyset$, we have that $\mu(\overline{D}) = \mu(\overline{C}_1) + \mu(\overline{D}_1) = \mu(\overline{C}_1) \leq \sum_{x \in C_1} \mu_1(D(x)) \mu_2(D(x)) \leq \sum_{x \in C_1} 2\mu_1(x) \frac{\varepsilon}{2} \leq \varepsilon$. ■

Some Comments on Theorem 5b of Dasgupta and Maskin (1986)

Consider a two-player game $G = (X_i, u_i)_{i \in \{1, 2\}}$ on the unit square, $X = X_1 \times X_2 = [0, 1] \times [0, 1]$, where the payoff functions' discontinuities lie on the main diagonal of X . Let $S^1 = \{(x_1, x_2) \in [0, 1] \times [0, 1] : x_2 > x_1\}$, $S^2 = \{(x_1, x_2) \in [0, 1] \times [0, 1] : x_1 > x_2\}$, and $S = \{(x_1, x_2) \in [0, 1] \times [0, 1] : x_1 = x_2\}$. Denote the set of points at which the sum of the payoff functions is not upper semicontinuous by S_D ; that is, $S_D = \{x \in X : \limsup_{y \rightarrow x} A_G^0(y) > A_G^0(x)\}$. The following statement is a slightly amended version of Theorem 5b of Dasgupta and Maskin (1986).

Theorem 7 Consider a two-player game $G = (X_i, u_i)_{i \in \{1, 2\}}$ on the unit

square $X = [0, 1] \times [0, 1]$. Assume that

- (i) there are continuous functions $f_i^j : \text{cl}S^j \rightarrow \mathbb{R}$, $i = 1, 2$, $j = 1, 2$ such that $u_i(x) = f_i^j(x)$ for all $x \in S^j$ and all $i, j \in \{1, 2\}$;
- (ii) for each $i \in \{1, 2\}$ and every $x \in S$, there exists $j \in \{1, 2\}$ such that

$$f_i^j(x) \geq u_i(x) \geq f_i^{-j}(x);$$

- (iii) for every point $x = (z, z) \in S_D$, there exist $i, j \in \{1, 2\}$ such that $\lim_{k \rightarrow \infty} f_i^j(x_i^k, z) > u_i(x)$ for some sequence $\{(x_i^k, z)\} \subset S^j$ converging to x ;
- (iv) if $f_i^j(x) > u_i(x)$ for some $x \in S_D$ and $i, j \in \{1, 2\}$, then $f_{-i}^j(x) < u_{-i}(x)$;
- (v) $A_G^0(x) \leq \max_{j \in \{1, 2\}} \{f_1^j(x) + f_2^j(x)\}$ for every $x \in S$.

Then G has a mixed strategy Nash equilibrium.

Condition (iii) is slightly stronger than the assumption that for every point $x = (z, z) \in S_D$, there exist $i, j \in \{1, 2\}$ such that $f_i^j(x) > u_i(x)$. The difference between them pertains to the points $(0, 0)$ and $(1, 1)$ only. The strengthening is needed to make it impossible for an equilibrium product measure of the modified game to have mass points at $(0, 0)$ or $(1, 1)$. Condition (v) must also be added to Theorem 5b. If $A_G^0(\tilde{x}) > \max_{j \in \{1, 2\}} \{f_1^j(\tilde{x}) + f_2^j(\tilde{x})\}$ for some $\tilde{x} \in S$ and A_G^0 is not upper semicontinuous at \tilde{x} along the main diagonal of the unit square, then the payoff-modifying technique may be ineffective for $\tilde{x} \in S$.

Finally, one can check that (i) and (ii) imply that both the initial and the modified games are uniformly payoff secure.

Proof of Theorem 5

For a given $\delta > 0$ and $z \in [0, 1]$, let $\tilde{C}_\delta(z)$ denote the set of all $z' \in [0, 1]$ such that $|z' - z| \leq \frac{\delta}{2}$. For $x = (x_1, x_2) \in [0, 1] \times [0, 1]$, let $C_\delta(x) = \tilde{C}_\delta(x_1) \times \tilde{C}_\delta(x_2)$. Denote by $\mathcal{B}_{[0,1]}$ the σ -algebra of Borel sets on $[0, 1]$. For every $\nu \in \Delta([0, 1])$ and a Borel set $Y \subset [0, 1]$, denote by $\nu|_Y$ the restriction of the probability measure ν to Y ; that is, $\nu|_Y(A) = \nu(Y \cap A)$ for every $A \in \mathcal{B}_{[0,1]}$. If $\nu(Y) > 0$,

then denote by $\bar{\nu}_{|Y}$ the probability measure defined by $\bar{\nu}_{|Y}(A) = \frac{\nu(Y \cap A)}{\nu(Y)}$ for every $A \in \mathcal{B}_{[0,1]}$.

We will show that every game on the unit square satisfying (i)-(iv) has a weakly reciprocally upper semicontinuous mixed extension. Consider a sequence of $\mu^k = (\mu_1^k, \mu_2^k) \in \Delta(X)$ converging weakly to $\hat{\mu} \in \Delta(X)$ such that $\lim_k U_i(\mu^k) > U_i(\hat{\mu})$ for some $i \in \{1, 2\}$ and $\lim_k U_{-i}(\mu^k)$ also exists. We have to show that there exist $m \in \{1, 2\}$ and $\mu'_m \in \Delta(X_m)$ such that $U_m(\mu'_m, \hat{\mu}_{-m}) > \lim_k U_m(\mu^k)$.

Since U_i jumps down at $\hat{\mu}$ along the sequence $\{\mu^k\}$, it must be the case that $\hat{\mu}(S) > 0$. Denote by $W(\hat{\mu})$ the set of mass points of $\hat{\mu}$ belonging to S . It is nonempty and at most countable. Moreover, $\hat{\mu}(S) = \sum_{y \in W(\hat{\mu})} \hat{\mu}(y)$ (see Lemma 5). Without loss of generality, $W(\hat{\mu}) = \{y^1, \dots, y^L\}$. Then, for some small enough $\hat{\delta}_1 > 0$, $C_{\hat{\delta}_1}(y^s) \cap C_{\hat{\delta}_1}(y^t) = \emptyset$ for all $s, t \in \{1, \dots, L\}$, $s \neq t$. Denote $y^s = (z^s, z^s)$, $s = 1, \dots, L$. Assume also that, for each $s \in \{1, \dots, L\}$ and every $z \in \tilde{C}_{\hat{\delta}_1}(z^s) \setminus \{z^s\}$, $\hat{\mu}_m(\{z\}) = 0$, $m = 1, 2$, since if it is not the case for some $s \in \{1, \dots, L\}$ and some $m \in \{1, 2\}$, non-zero masses for $\hat{\mu}_m$ on $\tilde{C}_{\hat{\delta}_1}(z_s) \setminus \{z_s\}$ can be made negligibly small by choosing a smaller $\hat{\delta}_1$. Assume also that $\lim_k U_m(\mu_m^k|_{\tilde{C}_{\hat{\delta}_1}(z_s)}, \mu_{-m}^k)$ exists for each $s \in \{1, \dots, L\}$ and each $m \in \{1, 2\}$; if not, consider a subsequence of μ^k possessing this property.

Let $y^1 \in W(\hat{\mu})$ be such that $\liminf_k \int_{C_\delta(y^1)} u_i d\mu^k - \int_{C_\delta(y^1)} u_i d\hat{\mu} > a > 0$ for every $\delta \in (0, \hat{\delta}_1]$. Then there exist $j_1 \in \{1, 2\}$ and a subsequence of $\{\mu^k\}$, denoted again by $\{\mu^k\}$, such that $f_i^{j_1}(y^1) > u_i(y^1)$ and $\lim_k \mu^k(C_{\hat{\delta}_1}(y^1) \cap S^{j_1}) > \hat{\mu}(C_{\hat{\delta}_1}(y^1) \cap S^{j_1})$. The reciprocal upper semicontinuity of G implies that $f_{-i}^{j_1}(y^1) < u_{-i}(y^1)$. Define

$$b_{-i} = \frac{1}{2}(u_{-i}(y^1) - f_{-i}^{j_1}(y^1))(\lim_k \mu^k(C_{\hat{\delta}_1}(y^1) \cap S^{j_1}) - \hat{\mu}(C_{\hat{\delta}_1}(y^1) \cap S^{j_1})).$$

Since all f_i^j , $i, j \in \{1, 2\}$, and $u_{i|S}$, $i \in \{1, 2\}$, are uniformly continuous on their respective compact domains $\text{cl}S^j$ and S , for every $\varepsilon > 0$ there exists $\delta(\varepsilon) \in (0, \hat{\delta}_1]$ such that, for all $i, j \in \{1, 2\}$, $|f_i^j(x') - f_i^j(x'')| < \frac{\varepsilon}{2}$ for all x'

and x'' in $\text{cl}S^j$ with $\|x' - x''\| < \delta(\varepsilon)$ and $|u_{i|S}(x') - u_{i|S}(x'')| < \frac{\varepsilon}{2}$ for all x' and x'' in S with $\|x' - x''\| < \delta(\varepsilon)$.

It follows from the uniform payoff security of G that, for each $s \in \{1, \dots, L\}$, there exists $d_{-i}(z_s, \frac{b_{-i}}{4s}) \in X_{-i}$ such that for every $x_i \in X_i$, $u_{-i}(d_{-i}(z_s, \frac{b_{-i}}{4s}), w_i) > u_{-i}(z^s, x_i) - \frac{b_{-i}}{4s}$ for all w_i in some neighborhood $\mathcal{N}_{X_i}(x_i)$ of x_i in X_i . Pick some $\tilde{\delta}_1 \in (0, \delta(\frac{b_{-i}}{4s})]$ such that $B_{X_i}(y^s, \tilde{\delta}_1) \subset \mathcal{N}_{X_i}(y^s)$ for all $s \in \{1, \dots, L\}$.

We want to show that there exist $d_{-i}^1 \in X_{-i}$ and $\delta_1 \in (0, \tilde{\delta}_1]$ such that $U_{-i}(d_{-i}^1, \hat{\mu}_i) > \lim_k U_{-i}(\bar{\mu}_{-i}^k |_{\tilde{C}_{\delta_1}(z^1)}, \mu_i^k) + \frac{b_{-i}}{2\hat{\mu}_{-i}(\tilde{C}_{\delta_1}(z^1))}$. Consider first the case where $y^1 \in X \setminus \{(0, 0) \cup (1, 1)\}$. Put $d_{-i}^1 = d_{-i}(z^1, \frac{b_{-i}}{4})$ and $\delta_1 = \tilde{\delta}_1$. Notice that $U_{-i}(d_{-i}^1, \hat{\mu}_i) = \int_{X_i \setminus \tilde{C}_{\delta_1}(z^1)} u_{-i}(d_{-i}^1, x_i) d\hat{\mu}_i + \int_{\tilde{C}_{\delta_1}(z^1)} u_{-i}(d_{-i}^1, x_i) d\hat{\mu}_i$. Since $u_{-i}(d_{-i}^1, x_i) > u_{-i}(x_{-i}, x_i) - \frac{b_{-i}}{2}$ for every $(x_{-i}, x_i) \in \tilde{C}_{\delta_1}(z^1) \times (X_i \setminus \tilde{C}_{\delta_1}(z^1))$, we have that

$$U_{-i}(d_{-i}^1, \hat{\mu}_i |_{X_i \setminus \tilde{C}_{\delta_1}(z^1)}) > U_{-i}(\bar{\mu}_{-i} |_{\tilde{C}_{\delta_1}(z^1)}, \hat{\mu}_i |_{X_i \setminus \tilde{C}_{\delta_1}(z^1)}) - \hat{\mu}_i(X_i \setminus \tilde{C}_{\delta_1}(z^1)) \frac{b_{-i}}{2}.$$

Since $\hat{\mu}_i(\partial \tilde{C}_{\delta_1}(z^1)) = \hat{\mu}_{-i}(\partial \tilde{C}_{\delta_1}(z^1)) = 0$, u_{-i} is continuous on $\tilde{C}_{\delta_1}(z^1) \times (X_i \setminus \tilde{C}_{\delta_1}(z^1))$, and the sequence $\{(\bar{\mu}_{-i}^k |_{\tilde{C}_{\delta_1}(z^1)}, \mu_i^k |_{X_i \setminus \tilde{C}_{\delta_1}(z^1)})\}$ converges weakly to $(\bar{\mu}_{-i} |_{\tilde{C}_{\delta_1}(z^1)}, \hat{\mu}_i |_{X_i \setminus \tilde{C}_{\delta_1}(z^1)})$, we have that

$$U_{-i}(\bar{\mu}_{-i} |_{\tilde{C}_{\delta_1}(z^1)}, \hat{\mu}_i |_{X_i \setminus \tilde{C}_{\delta_1}(z^1)}) = \lim_k U_{-i}(\bar{\mu}_{-i}^k |_{\tilde{C}_{\delta_1}(z^1)}, \mu_i^k |_{X_i \setminus \tilde{C}_{\delta_1}(z^1)}).$$

Therefore,

$$U_{-i}(d_{-i}^1, \hat{\mu}_i |_{X_i \setminus \tilde{C}_{\delta_1}(z^1)}) > \lim_k U_{-i}(\bar{\mu}_{-i}^k |_{\tilde{C}_{\delta_1}(z^1)}, \mu_i^k |_{X_i \setminus \tilde{C}_{\delta_1}(z^1)}) - \hat{\mu}_i(X_i \setminus \tilde{C}_{\delta_1}(z^1)) \frac{b_{-i}}{2}.$$

Since $u_{-i}(d_{-i}^1, x_i) > \sup_{y \in \tilde{C}_{\delta_1}(z^1) \times \tilde{C}_{\delta_1}(z^1)} u_{-i}(y) - \frac{b_{-i}}{2}$ for every $x_i \in \tilde{C}_{\delta_1}(z^1)$, we have that

$$U_{-i}(d_{-i}^1, \hat{\mu}_i |_{\tilde{C}_{\delta_1}(z^1)}) > \hat{\mu}_i(\tilde{C}_{\delta_1}(z^1)) \left(\sup_{y \in \tilde{C}_{\delta_1}(z^1) \times \tilde{C}_{\delta_1}(z^1)} u_{-i}(y) - \frac{b_{-i}}{2} \right).$$

It follows from the definition of b_{-i} that for some subsequence of $\{\mu^k\}$, denoted again by $\{\mu^k\}$, $\widehat{\mu}_1(\widetilde{C}_{\delta_1}(z^1))\widehat{\mu}_2(\widetilde{C}_{\delta_1}(z^1)) \sup_{y \in \widetilde{C}_{\delta_1}(z^1) \times \widetilde{C}_{\delta_1}(z^1)} u_{-i}(y) - \lim_k U_{-i}(\mu_{-i}^k | \widetilde{C}_{\delta_1}(z^1), \mu_i^k | \widetilde{C}_{\delta_1}(z^1)) > b_{-i}$, and, therefore,

$$\begin{aligned} & \widehat{\mu}_i(\widetilde{C}_{\delta_1}(z^1)) \sup_{y \in \widetilde{C}_{\delta_1}(z^1) \times \widetilde{C}_{\delta_1}(z^1)} u_{-i}(y) \\ & > \lim_k U_{-i}(\mu_{-i}^k | \widetilde{C}_{\delta_1}(z^1), \mu_i^k | \widetilde{C}_{\delta_1}(z^1)) + \frac{b_{-i}}{\widehat{\mu}_{-i}(\widetilde{C}_{\delta_1}(z^1))}. \end{aligned}$$

Then $U_{-i}(d_{-i}^1, \widehat{\mu}_i | \widetilde{C}_{\delta_1}(z^1)) > \lim_k U_{-i}(\frac{\mu_{-i}^k | \widetilde{C}_{\delta_1}(z^1)}{\widehat{\mu}_{-i}(\widetilde{C}_{\delta_1}(z^1))}, \mu_i^k | \widetilde{C}_{\delta_1}(z^1)) - \widehat{\mu}_i(\widetilde{C}_{\delta_1}(z^1)) \frac{b_{-i}}{2} + \frac{b_{-i}}{\widehat{\mu}_{-i}(\widetilde{C}_{\delta_1}(z^1))}$. Since $\lim_k \mu_{-i}^k(\widetilde{C}_{\delta_1}(z^1)) = \widehat{\mu}_{-i}(\widetilde{C}_{\delta_1}(z^1))$, we have that

$$\lim_k U_{-i}(\frac{\mu_{-i}^k | \widetilde{C}_{\delta_1}(z^1)}{\widehat{\mu}_{-i}(\widetilde{C}_{\delta_1}(z^1))}, \mu_i^k | \widetilde{C}_{\delta_1}(z^1)) = \lim_k U_{-i}(\overline{\mu}_{-i}^k | \widetilde{C}_{\delta_1}(z^1), \mu_i^k | \widetilde{C}_{\delta_1}(z^1)),$$

and, therefore,

$$\begin{aligned} U_{-i}(d_{-i}^1, \widehat{\mu}_i) & > \lim_k U_{-i}(\overline{\mu}_{-i}^k | \widetilde{C}_{\delta_1}(z^1), \mu_i^k) \\ & - (\widehat{\mu}_i(\widetilde{C}_{\delta_1}(z^1)) + \widehat{\mu}_i(X_i \setminus \widetilde{C}_{\delta_1}(z^1))) \frac{b_{-i}}{2} + \frac{b_{-i}}{\widehat{\mu}_{-i}(\widetilde{C}_{\delta_1}(z^1))}, \end{aligned}$$

which implies that $U_{-i}(d_{-i}^1, \widehat{\mu}_i) > \lim_k U_{-i}(\overline{\mu}_{-i}^k | \widetilde{C}_{\delta_1}(z^1), \mu_i^k) + \frac{b_{-i}}{2\widehat{\mu}_{-i}(\widetilde{C}_{\delta_1}(z^1))}$.

Consider the case where $y^1 = (0, 0)$. If $-i = 1$, then, by virtue of (iii), $j_1 = 1$, and, therefore, one can find a small enough $\delta_1 \in (0, \widetilde{\delta}_1]$ such that $U_1(\frac{\delta_1}{2}, \widehat{\mu}_2) > \lim_k U_1(\overline{\mu}_1^k | \widetilde{C}_{\delta_1}(0), \mu_2^k) \frac{b_1}{2\widehat{\mu}_1(\widetilde{C}_{\delta_1}(0))}$, with $d_1^1 = \frac{\delta_1}{2}$. Assume that $-i = 2$. If $j_1 = 1$, then the uniform payoff security condition for player 2's strategy 0 can be used to find suitable d_2^1 and δ_1 . If $j_1 = 2$ and $f_2^1(0, 0) \geq u_2(0, 0)$, then, for some $\delta_1 \in (0, \widetilde{\delta}_1]$, $U_2(\frac{\delta_1}{2}, \widehat{\mu}_1) > \lim_k U_2(\overline{\mu}_2^k | \widetilde{C}_{\delta_1}(0), \mu_1^k) \frac{b_2}{2\widehat{\mu}_2(\widetilde{C}_{\delta_1}(0))}$, with $d_2^1 = \frac{\delta_1}{2}$ in this case. If $j_1 = 2$ and $f_2^1(0, 0) < u_2(0, 0)$, the uniform payoff security condition for player 2's strategy 0 can be used to find suitable d_2^1 and δ_1 . The case with $y^1 = (1, 1)$ can be handled in a similar manner.

We now want to show that, for each $s \in \{2, \dots, L\}$, there exist $d_{-i}^s \in X_{-i}$ and $\delta_s \in (0, \tilde{\delta}_1]$ such that $U_{-i}(d_{-i}^s, \hat{\mu}_i) > \lim_k U_{-i}(\bar{\mu}_{-i}^k|_{\tilde{C}_{\delta_s}(z^s)}, \mu_i^k) - \frac{b_{-i}}{4s\hat{\mu}_{-i}(\tilde{C}_{\delta_s}(z^s))}$. Let $y^s \in X \setminus \{(0, 0) \cup (1, 1)\}$. If $u_{-i}(y^s) > \min\{f_{-i}^1(y^s), f_{-i}^2(y^s)\}$ or $u_{-i}(y^s) = f_{-i}^1(y^s) = f_{-i}^2(y^s)$, then it is possible to make use of the uniform payoff security condition for player $-i$'s strategy y_{-i}^s in order to find suitable $d_{-i}^s = d_{-i}(z^s, \frac{b_{-i}}{4s})$ and δ_s . If y^s is a point of discontinuity of u_{-i} and $u_{-i}(y^s) \leq f_{-i}^j(y^s) \leq f_{-i}^{-j}(y^s)$ for some $j \in \{1, 2\}$, then pick a small enough $\delta_s \in (0, \tilde{\delta}_1]$ such that $U_{-i}(z^s + (-1)^j \frac{\delta_s}{2}, \hat{\mu}_i) > \lim_k U_{-i}(\bar{\mu}_{-i}^k|_{\tilde{C}_{\delta_s}(z^s)}, \mu_i^k) - \frac{b_{-i}}{4s\hat{\mu}_{-i}(\tilde{C}_{\delta_s}(z^s))}$, with $d_{-i}^s = z^s + (-1)^j \frac{\delta_s}{2}$.

Assume that $y^s = (0, 0)$ and $-i = 2$. It follows from (iii) that $u_2(0, 0) \geq f_2^2(0, 0)$. If $f_2^1(0, 0) \geq u_2(0, 0)$, then, for a small enough $\delta_s \in (0, \tilde{\delta}_1]$, $U_2(\frac{\delta_s}{2}, \hat{\mu}_1) > \lim_k U_2(\bar{\mu}_1^k|_{\tilde{C}_{\delta_s}(0)}, \mu_1^k) - \frac{b_2}{4s\hat{\mu}_2(\tilde{C}_{\delta_s}(0))}$, with $d_2^s = \frac{\delta_s}{2}$. If $f_2^1(0, 0) < u_2(0, 0)$, then put $d_2^s = 0$ and pick a suitable $\delta_s \in (0, \tilde{\delta}_1]$.

Assume that $y^s = (0, 0)$ and $-i = 1$. By (iii), $u_1(0, 0) \leq f_1^2(0, 0)$. If, moreover, $f_1^1(0, 0) \leq f_1^2(0, 0)$, then, for some $\delta_s \in (0, \tilde{\delta}_1]$, $U_1(\frac{\delta_s}{2}, \hat{\mu}_2) > \lim_k U_1(\bar{\mu}_1^k|_{\tilde{C}_{\delta_s}(0)}, \mu_2^k) - \frac{b_1}{4s\hat{\mu}_1(\tilde{C}_{\delta_s}(0))}$, with $d_1^s = \frac{\delta_s}{2}$. If $f_1^1(0, 0) > f_1^2(0, 0)$, then it is important whether there is some subsequence of $\{\mu^k\}$, denoted again by $\{\mu^k\}$, such that $\lim_k \mu^k(C_{\tilde{\delta}_1}(0, 0) \cap S^1) > \hat{\mu}(C_{\tilde{\delta}_1}(0, 0) \cap S^1)$. If this is not the case, then it is possible to choose a small enough $\delta_s \in (0, \tilde{\delta}_1]$ such that $U_1(\frac{\delta_s}{2}, \hat{\mu}_2) > \lim_k U_1(\bar{\mu}_1^k|_{\tilde{C}_{\delta_s}(0)}, \mu_2^k) - \frac{b_1}{4s\hat{\mu}_1(\tilde{C}_{\delta_s}(0))}$, with $d_{-i}^s = \frac{\delta_s}{2}$. This is so because by choosing $\delta_s \in (0, \tilde{\delta}_1]$ close enough to 0, we can make $\hat{\mu}(C_{\delta_s}(0, 0) \cap S^1)$ arbitrarily small, and, moreover, G is compact and $\limsup_k \mu^k(C_{\delta_s}(0, 0) \cap S^1) \leq \hat{\mu}(C_{\delta_s}(0, 0) \cap S^1)$. However, if such a subsequence of $\{\mu^k\}$ exists, then, we have to turn to considering player 2's payoffs ($-i = 2$), with putting $y^1 = (0, 0)$ and $j_1 = 1$. A similar problem can not occur in this case if $y^s = (1, 1)$ for some $s \in \{2, \dots, L\}$ by virtue of (iii).

Then $\lim_k U_{-i}(\mu_{-i}^k, \mu_i^k) = \sum_{s=\overline{1, L}} \hat{\mu}_{-i}(\tilde{C}_{\delta_s}(z^s)) \lim_k U_{-i}(\bar{\mu}_{-i}^k|_{\tilde{C}_{\delta_s}(z^s)}, \mu_i^k) + U_{-i}(\hat{\mu}_{-i}|_{X_{-i} \setminus \cup_{s=\overline{1, L}} \tilde{C}_{\delta_s}(z^s)}, \hat{\mu}_i)$, and, as a result, it is not difficult to see that $\lim_k U_{-i}(\mu_{-i}^k, \mu_i^k) < \sum_{s=\overline{1, L}} \hat{\mu}_{-i}(\tilde{C}_{\delta_s}(z^s)) U_{-i}(d_{-i}^s, \hat{\mu}_i) + U_{-i}(\hat{\mu}_{-i}|_{X_{-i} \setminus \cup_{s=\overline{1, L}} \tilde{C}_{\delta_s}(z^s)}, \hat{\mu}_i)$.

Define $\mu'_{-i} \in \Delta(X_{-i})$ as follows:

$$\mu'_{-i} = \sum_{s=1, \overline{L}} \widehat{\mu}_{-i}(\widetilde{C}_{\delta_s}(z^s)) \delta_{d_{-i}^s} + \widehat{\mu}_{-i}|_{X_{-i} \setminus \cup_{s=1, \overline{L}} \widetilde{C}_{\delta_s}(z^s)}.$$

Then we have that $\lim_k U_{-i}(\mu_{-i}^k, \mu_i^k) < U_{-i}(\mu'_{-i}, \widehat{\mu}_i)$ which implies that Γ is weakly reciprocally upper semicontinuous.

Proof of Theorem 6

For the sake of notational simplicity, denote $X_{h_1}^1$ by T^1 , $X_{h_1}^2 \cap X_{h_2}^1$ by T^2 , and $X_{h_2}^2$ by T^3 . Then, for each $i \in \{1, 2\}$, $u_i(x) = f_i^1(x)$ for every $x \in T^1$, $u_i(x) = f_i^2(x)$ for every $x \in T^2$, and $u_i(x) = f_i^3(x)$ for every $x \in T^3$.

Consider a sequence of $\mu^k = (\mu_1^k, \mu_2^k) \in \Delta(X)$ converging weakly to $\widehat{\mu} \in \Delta(X)$ such that $\lim_k U_i(\mu^k) > U_i(\widehat{\mu})$ for some $i \in \{1, 2\}$. Assume, without loss of generality, that $\lim_k U_{-i}(\mu^k)$ also exists. We have to show that there exist $m \in \{1, 2\}$ and $\mu'_m \in \Delta(X_m)$ such that $U_m(\mu'_m, \widehat{\mu}_{-m}) > \lim_k U_m(\mu^k)$. Since U_i jumps down at $\widehat{\mu}$ along the sequence $\{\mu^k\}$, it must be the case that $\widehat{\mu}(X_{h_1} \cup X_{h_2}) > 0$. Denote by $W(\widehat{\mu})$ the set of mass points of $\widehat{\mu}$ belonging to $X_{h_1} \cup X_{h_2}$. It is nonempty and at most countable. Moreover, $\widehat{\mu}(X_{h_1} \cup X_{h_2}) = \sum_{y \in W(\widehat{\mu})} \widehat{\mu}(y)$. Without loss of generality, $W(\widehat{\mu}) = \{y^1, \dots, y^L\}$, where $y^s = (y_1^s, y_2^s)$ for each $s \in \{1, \dots, L\}$. Then there exists some $\widehat{\delta}_1 > 0$ such that $C_{\widehat{\delta}_1}(y^s) \cap C_{\widehat{\delta}_1}(y^t) = \emptyset$ for all $s, t \in \{1, \dots, L\}$, $s \neq t$. Without loss of generality, for each $s \in \{1, \dots, L\}$ and each $m \in \{1, 2\}$, $\widehat{\mu}_m(\{z\}) = 0$ for every $z \in \widetilde{C}_{\widehat{\delta}_1}(z_m^s) \setminus \{z_m^s\}$. Assume also that $\lim_k U_m(\mu_m^k|_{\widetilde{C}_{\widehat{\delta}_1}(z_m^s)}, \mu_{-m}^k)$ exists for each $s \in \{1, \dots, L\}$ and each $m \in \{1, 2\}$. Without loss of generality, $y^1 = (z_1^1, z_2^1) \in X_{h_2}$, and, moreover, $\liminf_k \int_{C_\delta(y^1)} u_i d\mu^k - \int_{C_\delta(y^1)} u_i d\widehat{\mu} > a > 0$ for every $\delta \in (0, \widehat{\delta}_1]$.

For some $j_1 \in \{2, 3\}$ and some subsequence of $\{\mu^k\}$, denoted again by $\{\mu^k\}$, we have that $f_i^{j_1}(y^1) > u_i(y^1)$ and $\lim_k \mu^k(C_{\widehat{\delta}_1}(y^1) \cap T^{j_1}) > \widehat{\mu}(C_{\widehat{\delta}_1}(y^1) \cap T^{j_1})$. The reciprocal upper semicontinuity of G implies that $f_{-i}^{j_1}(y^1) <$

$u_{-i}(y^1)$. Define

$$b_{-i} = \frac{1}{2}(u_{-i}(y^1) - f_{-i}^{j_1}(y^1))(\lim_k \mu^k(C_{\widehat{\delta}_1}(y^1) \cap T^{j_1}) - \widehat{\mu}(C_{\widehat{\delta}_1}(y^1) \cap T^{j_1})).$$

Consider first the case where $-i = 2$. We will show that there exists $\mu'_2 \in \Delta(X_2)$ such that $U_2(\mu'_2, \widehat{\mu}_1) > \lim_k U_2(\mu_2^k, \mu_1^k)$. Since the details of the reasoning in this case is similar to those that can be found in the proof of Theorem 5, we will describe how to choose $d_2^1 \in X_2$ and $\delta_1 \in (0, \widehat{\delta}_1]$ such that $U_2(d_2^1, \widehat{\mu}_1) > \lim_k U_2(\overline{\mu}_2^k |_{\widetilde{C}_{\delta_1}(z_2^1)}, \mu_1^k) + \frac{b_2}{2\widehat{\mu}_2(\widetilde{C}_{\delta_1}(z_2^1))}$ in a schematic way.

Let $y^1 = (0, \frac{1}{4})$. If $j_1 = 2$ and $u_2(0, \frac{1}{4}) > f_2^3(0, \frac{1}{4})$ or $j_1 = 3$ and $u_2(0, \frac{1}{4}) > f_2^2(0, \frac{1}{4})$, then one can use the uniform payoff security condition for player 2's strategy $\frac{1}{4}$ to find suitable d_2^1 and δ_1 . If $j_1 = 3$ and $u_2(0, \frac{1}{4}) \leq f_2^2(0, \frac{1}{4})$ or $j_1 = 2$ and $u_2(0, \frac{1}{4}) \leq f_2^3(0, \frac{1}{4})$, then $d_2^1 = \frac{1}{4} + (-1)^{j_1} \frac{\delta_1}{2}$ with a small enough $\delta_1 \in (0, \widehat{\delta}_1]$. The cases where $y^1 \in X_{h_2} \setminus \{(0, \frac{1}{4}) \cup (1, 0)\}$ can be handled in a similar manner.

Assume that $y^1 = (1, 0)$. By virtue of (iii), $u_2(1, 0) \geq f_2^3(1, 0)$. If $j_1 = 2$ and $u_2(1, 0) > f_2^3(1, 0)$ or $j_1 = 3$ and $u_2(1, 0) > f_2^2(1, 0)$, then the uniform payoff security condition for player 2's strategy 0 can be applied. If $j_1 = 2$ and $u_2(1, 0) = f_2^3(1, 0)$ or $j_1 = 3$ and $u_2(1, 0) \leq f_2^2(1, 0)$, then $d_2^1 = (j_1 - 2) \frac{\delta_1}{2}$ with a small enough $\delta_1 \in (0, \widehat{\delta}_1]$. The cases where $y^1 \in X_{h_2} \setminus (\{(0, \frac{1}{4})\} \cup \{(1, 0)\})$ can be handled with no difficulty through making use of the uniform payoff security condition for player 2's strategy y_2^1 .

We now have to describe how to choose, for each $s \in \{2, \dots, L\}$, $d_2^s \in X_2$ and $\delta_s \in (0, \widehat{\delta}_1]$ such that $U_2(d_2^s, \widehat{\mu}_1) > \lim_k U_2(\overline{\mu}_2^k |_{\widetilde{C}_{\delta_s}(z_2^s)}, \mu_1^k) - \frac{b_2}{4s\widehat{\mu}_2(\widetilde{C}_{\delta_s}(z_2^s))}$. It is also done in a schematic way.

Assume, for example, that $y^s = (0, \frac{1}{4})$ for some $s \in \{2, \dots, L\}$. If $u_2(0, \frac{1}{4}) \geq \max\{f_2^2(0, \frac{1}{4}), f_2^3(0, \frac{1}{4})\}$, then the uniform payoff security condition for player 2's strategy $\frac{1}{4}$ can be used to find suitable d_2^s and δ_s . Let $u_2(0, \frac{1}{4}) < \max\{f_2^2(0, \frac{1}{4}), f_2^3(0, \frac{1}{4})\}$. If $f_2^2(0, \frac{1}{4}) \geq f_2^3(0, \frac{1}{4})$ ($f_2^2(0, \frac{1}{4}) < f_2^3(0, \frac{1}{4})$), then $d_2^s = \frac{1}{4} + \frac{\delta_s}{2}$ ($d_2^s = \frac{1}{4} - \frac{\delta_s}{2}$) with a small enough $\delta_s \in (0, \widehat{\delta}_1]$. The cases

where $y^s \in X_{h_2} \setminus \{(0, \frac{1}{4})\}$ for some $s \in \{2, \dots, L\}$ can be handled in a similar manner. The only case to be mentioned separately is where $y^s = (1, 0)$ and $f_2^2(1, 0) < f_2^3(1, 0)$. Recalling that $u_2(1, 0) \geq f_2^3(1, 0)$ by virtue of (iii), put $d_2^s = 0$ and pick a small enough $\delta_s \in (0, \widehat{\delta}_1]$. A similar argument can be provided for the mass points of $\widehat{\mu}_2$ lying on X_{h_1} .

Thus, if, for some $s \in \{1, \dots, L\}$, some $j_1 \in \{1, 2, 3\}$, and some subsequence of $\{\mu^k\}$, denoted again by $\{\mu^k\}$, the inequalities $f_1^{j_1}(y^s) > u_1(y^s)$ and $\lim_k \mu^k(C_{\widehat{\delta}_1}(y^s) \cap T^{j_1}) > \widehat{\mu}(C_{\widehat{\delta}_1}(y^s) \cap T^{j_1})$ hold, then it is possible to construct $\mu'_2 \in \Delta(X_2)$ such that $U_2(\mu'_2, \widehat{\mu}_1) > \lim_k U_2(\mu_2^k, \mu_1^k)$. Assume now this is not the case; that is, $-i = 1$. We shall show that there exists $\mu'_1 \in \Delta(X_1)$ such that $U_1(\mu'_1, \widehat{\mu}_2) > \lim_k U_1(\mu_1^k, \mu_2^k)$. If $j_1 = 2$ and y^1 is $(1, 0)$ or $(1, 1)$, then put $d_1^1 = 1 - \frac{\delta_1}{2}$ with a small enough $\delta_1 \in (0, \widehat{\delta}_1]$. If $y^1 \in (X_{h_1} \cup X_{h_2}) \setminus (\{(1, 0)\} \cup \{(1, 1)\})$, then the uniform payoff security condition for player 1's strategy 1 can be used to find suitable d_1^1 and δ_1 .

If, for some $s \in \{2, \dots, L\}$, $y^s = (1, 0)$ (or $(1, 1)$), then $d_1^s = 1 - \frac{\delta_s}{2}$ with a small enough $\delta_s \in (0, \widehat{\delta}_1]$, which is possible owing to the fact that $-i$ can not be equal to 2. Let $y^s = (y_1^s, y_2^s) \in X_{h_2} \setminus \{(1, 0)\}$ for some $s \in \{2, \dots, L\}$. Denote by $y^d = (y_1^d, y_2^d)$ the point belonging to X_{h_1} such that $y_1^d = y_1^s$ and $y_2^d \neq y_2^s$. If $u_1(y_s) \leq f_1^2(y_s)$ and $u_1(y_d) \leq f_1^2(y_d)$, then $d_1^s = y_1^s + \frac{\delta_s}{2}$ with a small enough $\delta_s \in (0, \widehat{\delta}_1]$, which is possible since if $f_1^3(y_s) > u_1(y_s)$ ($f_1^1(y_d) > u_1(y_d)$), then it must be the case that $\limsup_k \mu^k(C_{\delta_s}(y_s) \cap T^3) \leq \widehat{\mu}(C_{\delta_s}(y_s) \cap T^3)$ ($\limsup_k \mu^k(C_{\delta_s}(y_d) \cap T^1) \leq \widehat{\mu}(C_{\delta_s}(y_d) \cap T^1)$). If $u_1(y_s) > f_1^2(y_s)$ or $u_1(y_s) > f_1^2(y_s)$, then the uniform payoff security condition for player 1's strategy y_1^s can be used to find suitable d_1^s and δ_s .

References

Aliprantis, C.D, Border, K.C., 2006. Infinite Dimensional Analysis, A Hitchhiker's Guide, 3rd ed. Berlin: Springer-Verlag.

- Amir, R., 2005. Ordinal versus cardinal complementarity: The case of Cournot oligopoly. *Games and Econ. Behav.* 53, 1-14.
- Bagh, A., Jofre, A., 2006. Reciprocal upper semicontinuity and better reply secure games: a comment. *Econometrica* 74, 1715–1751.
- Ball, R., 1999. Discontinuity and nonexistence of equilibrium in the probabilistic spatial voting model. *Soc. Choice Welfare* 16, 533-555.
- Barelli, P., Meneghel, I., 2013. A note on the equilibrium existence problem in discontinuous games, *Econometrica* 81, 813-824.
- Baye, M.R., Tian, G., Zhou, J., 1993. Characterization of the existence of equilibria in games with discontinuous and non-quasiconcave payoffs. *Rev. Econ. Stud.* 60, 935-948.
- Baye, M.R., Kovenock, D., de Vries, C.G., 1994. The solution to the Tullock rent-seeking game when $R > 2$: Mixed-strategy equilibria and mean dissipation rates. *Public Choice* 81, 363-380.
- Baye, M.R., Kovenock, D., de Vries, C.G., 1996. The all-pay auction with complete information. *Econ. Theory* 8, 291-305.
- Bich, P., 2009. Existence of pure Nash equilibria in discontinuous and non quasiconcave games. *Int. J. Game Theory* 38, 395-410.
- Carbonell-Nicolau, O., Ok, E.A., 2007. Voting over income taxation. *J. Econ. Theory* 134, 249-286.
- Carmona, G., 2005. On the existence of equilibria in discontinuous games: three counterexamples. *Int. J. Game Theory* 33, 181-187.
- Carmona, G., 2011. Understanding some recent existence results for discontinuous games. *Econ. Theory* 48, 31-45.
- de Castro, L.I., 2011. Equilibria existence in regular discontinuous games, *Econ. Theory* 48, 67-85.
- Dasgupta, P., Maskin, E., 1986. The existence of equilibrium in discontinuous economic games, I: Theory. *Rev. Econ. Stud.* 53, 1-26.
- Ding, X.P., Park, J.Y., 2002. Continuous selection theorem, coincidence theorem, and generalized equilibrium in L -convex spaces. *Comput. Math.*

Appl. 44, 95-103.

Duggan, J., 2007. Equilibrium existence for zero-sum games and spatial models of elections. *Games and Econ. Behav.* 60, 52-74.

Glicksberg, I.L., 1952. A Further generalization of the Kakutani fixed point theorem with applications to Nash equilibrium points. *Proc. Amer. Math. Soc.* 38, 170–174

Glycopantis, D., Muir, A., 2004. The compactness of $\text{Pr}(K)$. *Adv. Math. Econ.* 6, 39-53.

Lan, K.Q., Wu, J.H., 2002. A fixed-point theorem and applications to problems on sets with convex sections and to Nash equilibria. *Math. Comput. Model* 36, 139-145.

McLennan, A., Monteiro, P.K., Tourky, R., 2011. Games with discontinuous payoffs: a strengthening of Reny’s existence theorem. *Econometrica* 79, 1643-1664.

Monteiro, P.K., Page, F.H., 2007. Uniform payoff security and Nash equilibrium in compact games. *J. Econ. Theory* 134, 566-575.

Page, F.H., Monteiro, P.K., 2003. Three principles of competitive non-linear pricing. *J. Math. Econ.* 39, 63-109.

Prokopovych, P., 2011. On equilibrium existence in payoff secure games. *Econ. Theory* 48, 5-16.

Prokopovych, P., 2013. The single deviation property in games with discontinuous payoffs. *Econ. Theory* 53, 383-402.

Reny, P.J., 1999. On the existence of pure and mixed strategy Nash equilibria in discontinuous games. *Econometrica* 67, 1029-1056.

Reny, P.J., 2009. Further results on the existence of Nash equilibria in discontinuous games. Manuscript, University of Chicago.

Reny, P.J., 2011. Strategic approximation of discontinuous games. *Econ. Theory* 48, 17-29.

Simon, L., 1987. Games with discontinuous payoffs. *Rev. Econ. Stud.* 54, 569-597.

Sion, M., Wolfe, P., 1957. On a game without a value, in: Contributions to the Theory of Games, III, Princeton. Ann. Math. Stud. 39, 299-306.

Tian, G., 1992. Generalizations of the FKKM theorem and the Ky Fan minimax Inequality, with applications to maximal elements, price equilibrium, and complementarity. J. Math. Anal. Appl. 170, 457-471.

Tullock, G., 1980. Efficient rent-seeking, In: J.M. Buchanan et al. (Eds) Toward a Theory of the Rent-Seeking Society, Texas A&M University Press, College Station.

Yannelis, N.C., Prabhakar, N.D., 1983. Existence of maximal elements and equilibria in linear topological spaces. J. Math. Econ. 12, 233-245.

Yannelis, N.C., 1991. The core of an economy without ordered preferences, In: Khan, M.A., Yannelis, N.C. (eds.) Equilibrium Theory in Infinite Dimensional Spaces. Springer, Berlin.