Majorized correspondences and equilibrium existence in discontinuous games[1]

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Abstract This paper is aimed at widening the scope of applications of majorized correspondences. A new class of majorized correspondences – domain $\mathcal{U}$-majorized correspondences – is introduced. For them, a maximal element existence theorem is established. Then, sufficient conditions for the existence of an equilibrium in qualitative games are provided. They are used to show the existence of a pure strategy Nash equilibrium in compact quasiconcave games that are either correspondence secure or correspondence transfer continuous.

Keywords Majorized correspondence; Qualitative game; Better-reply secure game; Correspondence secure game; Transfer continuous game

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1 Introduction

In spite of the fact that the foundations of the theory of majorized correspondences were laid down by economists (Borglin and Keiding 1976; Yannelis and Prabhakar 1983), it has become, over time, mathematicians' ponderous object of study, with applications never reaching normal form games. Recent progress in equilibrium existence theory has considerably revived interest in using nonlinear analysis tools among economists. Of late, a number of generalizations of Reny's (1999) equilibrium existence theorem have been developed: McLennan, Monteiro, and Tourky (2011) proposed to use multiple restrictional security in place of Reny's better-reply security and quasiconcavity; Barelli and Meneghel (2013) introduced continuous security; and Reny (2013) generalized better-reply security to correspondence security.[3] The goal of this paper is to look at these and some other recent contributions to equilibrium existence theory through the prism of majorized correspondences, which will allow us not only to make some proofs more straightforward but also to relax some equilibrium existence conditions.

It may be too cumbersome, in theoretical assumptions, to demand that a correspondence have open lower sections because this property is stronger than lower hemicontinuity. However, correspondences with open lower sections have found important applications in existence theory, owing to the fact that they have been used for majorizing less well-behaved correspondences (see, e.g., Yannelis and Prabhakar 1983; Ding and Tan 1993; Yuan and Tarafdar 1996; Ding and Yuan 1998). More recently, Prokopovych (2011) proposed a short proof of Reny's (1999) equilibrium existence theorem for payoff secure games by employing the fact that, in such games, the approximate best-reply correspondences have multivalued selections with open lower sections.[4] Then, to facilitate studying equilibrium existence in discontinuous

[3] Some other recent equilibrium existence results can be found in Bich (2009), Carbonell-Nicholau and Ok (2007), de Castro (2011), Duggan (2007), Monteiro and Page (2007), Prokopovych and Yannelis (2014), and Reny (2011).

[4] Related theoretical results can be found in Baye, Tian, and Zhou (1993), Tian (1992),

games, Prokopovych (2013) introduced the notion of a domain L-majorized correspondence, which is, to some extent, related to Yuan's (1999) notion of an F_θ-majorized correspondence. The idea behind a domain majorized correspondence is that there exists a better-behaved correspondence whose domain contains the former correspondence's domain.

Another, less popular branch of the literature studying majorized correspondences employs, as better-behaved, not correspondences with open lower sections but upper hemicontinuous correspondences (see, e.g., Tan and Yuan 1993; Ding 1998; Yuan and Tarafdar 1999). From an applications' point of view, it is often preferable to use $\mathcal{U}$-majorized correspondences, not L-majorized, because it has become conventional for game theorists to select players' deviation strategies from the values of some upper hemicontinuous correspondences (see, e.g., Barelli and Soza 2013; Carmona 2011; Nessah 2011; Reny 2013; Scalzo 2013). Consequently, by a well-behaved correspondence, it is usually meant either an upper hemicontinuous correspondence or even a correspondence that has an upper hemicontinuous multivalued selection (see, e.g., Carmona 2014; He and Yannelis 2014).[5] According to Carmona (2014), a correspondence is a reducible fixed point correspondence if there exists a well-behaved correspondence whose set of fixed points is a subset of the set of fixed points of the correspondence under study. This approach is, in a certain sense, dual to the approach based on majorized correspondences, since, in the latter, the objective is to show that the set of maximal elements of a better-behaved correspondence is contained in the set of maximal elements of the correspondence under study.

The structure of this paper is as follows. First, we introduce the notion of a domain $\mathcal{U}$-majorized correspondence, called a $\mathcal{U}^d$-majorized correspondence. Then we provide a maximal element existence theorem for $\mathcal{U}^d$-majorized correspondences, Theorem 1, and some sufficient conditions

and Yannelis (1991).

[5] See also Carmona (2009) where a class of games with upper hemicontinuous best-reply correspondences is studied.

for a correspondence to be $\mathcal{U}^d$-majorized, Lemmas 3 and 4. In Section 3, equilibrium existence in qualitative games is studied. The focus lies not on generalizing some known results on $\mathcal{U}-$majorized correspondences, but on developing a theoretical background for studying equilibrium existence in normal-form games. In Section 4, Corollary 2 and a more advanced statement, Theorem 2, from Section 3 are used to show the existence of an equilibrium in compact quasiconcave games that are correspondence secure and correspondence transfer continuous, respectively.[6] Some technical proofs are postpone to the Appendix.

2 Domain U-Majorized Correspondences

In this section, we provide some definitions and facts about $\mathcal{U}$-majorized correpondences. A number of basic properties of $\mathcal{U}^d$-majorized correpondences are studied.

Let X be a topological space, Y be a nonempty convex subset of a topological vector space, $\theta : X \to Y$ be a single-valued function, and $F : X \twoheadrightarrow Y$ be a correspondence. The domain of F is defined by $\mathrm{Dom}F = \{x \in X : F(x) \neq \varnothing\}$; its graph is given by $\mathrm{Gr}F = \{(x, y) \in X \times Y : y \in F(x)\}$. If $\mathrm{Dom}F = X$, then F is strict; if $\mathrm{Gr}F$ is a closed subset of $X \times Y$, then F is closed. It is said that

(1) F is upper hemicontinuous at $x \in X$ if for any open subset U of Y containing $F(x)$, the set $\{z \in X : F(z) \subset U\}$ is an open neighborhood of x in X; F is upper hemicontinuous on X if F is upper hemicontinuous at x for every $x \in X$;

(2) F is of class $\mathcal{U}_\theta$ if F is upper hemicontinuous with closed and convex values in Y and $\theta(x) \notin F(x)$ for every $x \in X$;

[6] The notion of a correspondence transfer continuous game, introduced in this paper, is an improvement on the notion of a generalized weakly transfer continuous game, due to Nessah (2011).

(3) F^x is $\mathcal{U}_\theta$-majorant of F at x if for some open neighborhood $U(x)$ of x in X (endowed with the relative topology from X), $F^x : U(x) \twoheadrightarrow Y$ satisfies the following two properties: (a) $F(z) \subset F^x(z)$ for every $z \in U(x)$ and (b) F^x is of class $\mathcal{U}_\theta$ on $U(x)$;

(4) F is locally $\mathcal{U}_\theta$-majorized if for every $x \in \mathrm{Dom}F$, there exists a $\mathcal{U}_\theta$-majorant of F at x;

(5) F is $\mathcal{U}_\theta$-majorized if there exists a correspondence $\overline{F} : X \twoheadrightarrow Y$ of class $\mathcal{U}_\theta$ such that $F(x) \subset \overline{F}(x)$ for every $x \in X$.

In the special cases when $Y = X$ and θ is the identity map on X, and when $X = \Pi_{i=1}^n X_i$, $Y = X_i$, and $\theta : X \to X_i$ is the projection of X onto X_i, we will write $\mathcal{U}$ in place of $\mathcal{U}_\theta$. For a subset B of a topological vector space, denote the closure of B by $\mathrm{cl}B$, and the convex hull of B by $\mathrm{co}B$.

The next maximal element existence result is equivalent to the Kakutani-Fan-Glicksberg fixed point theorem.

Lemma 1 *Let X be a nonempty convex compact subset of a locally convex Hausdorff space, and let $F : X \twoheadrightarrow X$ be a correspondence of class $\mathcal{U}$. Then there exists $\widehat{x} \in X$ such that $F(\widehat{x}) = \varnothing$.*

Lemma 2 states that there are no differences between $\mathcal{U}_\theta$-majorized and locally $\mathcal{U}_\theta$-majorized correspondences from the standpoint of the applications we are interested in (see, e.g., Tan and Yuan 1993, Theorem 2.1; and Yannelis and Prabhakar 1983, Corollary 5.1 for L-majorized correspondences). For the reader's convenience, its proof is given in the Appendix.

Lemma 2 *Let X be a compact topological space, Y be a nonempty convex compact subset of a Hausdorff topological vector space. Let $\theta : X \to Y$ and $F : X \twoheadrightarrow Y$ be a strict correspondence. Then F is locally $\mathcal{U}_\theta$-majorized if and only if it is $\mathcal{U}_\theta$-majorized.*

Corollary 1 follows from Lemmas 1 and 2 by way of contradiction.

Corollary 1 *Let X be a nonempty convex compact subset of a locally convex Hausdorff space and $F : X \twoheadrightarrow X$ be a locally $\mathcal{U}$-majorized correspondence. Then there exists $\widehat{x} \in X$ such that $F(\widehat{x}) = \varnothing$.*

We now introduce a generalization of the notion of a $\mathcal{U}$-majorized correspondence.

Definition 1 *Let X be a topological space, Y be a nonempty convex subset of a topological vector space, and let $\theta : X \to Y$. A correspondence $F : X \twoheadrightarrow Y$ is $\mathcal{U}_\theta^d$-majorized if there exists a correspondence $\overline{F} : X \twoheadrightarrow Y$ of class $\mathcal{U}_\theta$ such that $\mathrm{Dom}F \subset \mathrm{Dom}\overline{F}$.*

The superindex d in $\mathcal{U}_\theta^d$ indicates the fact that F is domain majorized by $\overline{F}$. Clearly, if F is $\mathcal{U}_\theta$-majorized, then it is $\mathcal{U}_\theta^d$-majorized.

Theorem 1 is a maximal element existence theorem for $\mathcal{U}^d$-majorized correspondences.

Theorem 1 *Let X be a nonempty convex compact subset of a locally convex Hausdorff space, and let $F : X \twoheadrightarrow X$ be a $\mathcal{U}^d$-majorized correspondence. Then there exists $\widehat{x} \in X$ such that $F(\widehat{x}) = \varnothing$.*

Proof. Since $F : X \twoheadrightarrow X$ is $\mathcal{U}_\theta^d$-majorized, there is a correspondence $\overline{F} : X \twoheadrightarrow X$ of class $\mathcal{U}_\theta$ such that $\mathrm{Dom}F \subset \mathrm{Dom}\overline{F}$. Then, by Lemma 1, $\overline{F}(\widehat{x}) = \varnothing$ for some $\widehat{x} \in X$, which implies that $F(\widehat{x}) = \varnothing$. ■

The next lemma provides a set of sufficient conditions for a correspondence to be $\mathcal{U}_\theta^d$-majorized. For a set X, let $\langle X \rangle$ denote the family of its nonempty finite subsets.

Lemma 3 *Let X be a compact Hausdorff topological space and Y be a nonempty convex compact subset of a Hausdorff topological vector space. Let $\theta : X \to Y$ and $F : X \twoheadrightarrow Y$ be a strict correspondence. If for each $x \in X$, there exist an open neighborhood $U(x)$ of x in X and an upper hemicontinuous correspondence $F^x : U(x) \twoheadrightarrow Y$ with nonempty convex compact values* *such that for each $A \in \langle X \rangle$ and every $z \in \cap_{x\in A} U(x)$, $\theta(z) \notin \text{co}\{\cup_{x\in A} F^x(z)\}$, then F is $\mathcal{U}^d_\theta$-majorized.*

Proof. Since X is compact, the open cover $\{U(x) : x \in X\}$ of X contains a finite subcover $\{U(x_j) : j \in J\}$, where J is a finite set. Let $\{C_j : j \in J\}$ be a closed refinement of $\{U(x_j) : j \in J\}$ such that $C_j \subset U(x_j)$ for each $j \in J$. For each $j \in J$, define a correspondence $F^0_j : X \twoheadrightarrow Y$ by

$$F^0_j(z) = \begin{cases} \cup_{\{s\in J: z\in C_s\}} F^{x_s}(z) \text{ if } z \in U(x_j), \\ Y \text{ if } z \notin U(x_j). \end{cases}$$

Let W be a relatively open subset of Y and let $F^0_j(z) \subset W$ for some $z \in U_j$. We need to show that there exists a neighborhood U of z such that $F^0_j(U) \subset W$. Denote $J^z_C = \{s \in J : z \in C_s\}$. It follows from the upper hemicontinuity of the F^{x_s}'s that, for every $s \in J_z$, $F^{x_s}(U^0_s) \subset W$ for some open neighborhood $U^0_s \subset U(x_s)$. Therefore, $F^0_j(U) \subset W$ for $U = (X \backslash (\cup_{s\in J\backslash J^z_C} C_s)) \cap (\cap_{s\in J^z_C} U^0_s)$.

Define $F_j : X \twoheadrightarrow Y$ by $F_j(z) = \text{co}F^0_j(z)$ for every $z \in X$ and each $j \in J$. Since the convex hull of a finite union of convex compact sets is compact (see, e.g., Aliprantis and Border 2006, Lemma 5.29), each F_j is compact-valued. Therefore, each F_j is also upper hemicontinuous on X (see, e.g., Aliprantis and Border 2006, Theorem 17.35). Clearly, $\theta(z) \notin F_j(z)$ for all $z \in U(x_j)$ and $j \in J$. Then, $\overline{F} : X \twoheadrightarrow Y$ defined by $\overline{F}(z) = \cap_{j\in J} F_j(z)$ is upper hemicontinuous since Y is a compact Hausdorff space (see, e.g., Aliprantis and Border 2006, Theorem 17.25); that is, $\overline{F}$ is of class $\mathcal{U}_\theta$. By construction, $\text{Dom}\overline{F} = X$. ■

Another set of sufficient conditions obtains if the condition that, for each $A \in \langle X \rangle$ and every $z \in \cap_{x\in A} U(x)$, $\theta(z) \notin \text{co}\{\cup_{x\in A} F^x(z)\}$ is replaced with the condition that $\cap_{x\in A} U(x) \subset \text{Dom}(\cap_{x\in A} F_x)$ for each $A \in \langle X \rangle$, which is present, for example, in the definition of an F_θ-majorized correspondence (see, Yuan

1999, p. 411). Here the correspondence $(\cap_{x\in A}F^x) : \cap_{x\in A}U(x) \twoheadrightarrow Y$ is defined by $(\cap_{x\in A}F^x)(z) = \cap_{x\in A}F^x(z)$ for every $z \in \cap_{x\in A}U(x)$.

Lemma 4 *Let X be a compact Hausdorff topological space and Y be a nonempty convex compact subset of a Hausdorff topological vector space. Let $\theta : X \to Y$ and $F : X \twoheadrightarrow Y$ be a strict correspondence. If for each $x \in X$, there exist an open neighborhood $U(x)$ of x in X and an upper hemicontinuous correspondence $F^x : U(x) \twoheadrightarrow Y$ with nonempty convex compact values such that for each $A \in \langle X\rangle$, $\cap_{x\in A}U(x) \subset \mathrm{Dom}(\cap_{x\in A}F^x)$, then F is $\mathcal{U}_\theta^d$-majorized.*

The proof of Lemma 4 is given in the Appendix.

3 Qualitative Games

This section's two equilibrium existence results – Corollary 2 and Theorem 2 – are used in Section 4 for studying the existence of a pure strategy Nash equlibrium in strategic games.

Let I denote a set of players (countable or uncountable). Each player i's strategy set X_i is a nonempty, compact, and convex subset of a locally convex Hausdorff space. Let $X = \Pi_{i\in I}X_i$, and let $P_i : X \twoheadrightarrow X_i$ denote player i's preference correspondence. This correspondence may also be called player i's better-reply correspondence.

Consider a qualitative game $\Gamma = (X_i, P_i)_{i\in I}$. A strategy profile $x \in X$ is an equilibrium of Γ if $P_i(x) = \varnothing$ for all $i \in I$; in other words, all the better-reply correspondences are empty-valued at x.

For a qualitative game $\Gamma = (X_i, P_i)_{i\in I}$, we call the set $\mathrm{Dom}\Gamma = \cup_{i\in I}\mathrm{Dom}P_i$ the domain of Γ.

Definition 2 *A game $\Gamma = (X_i, P_i)_{i\in I}$ is $\mathcal{U}^d$-majorized if there exists a correspondence $\overline{F} : X \twoheadrightarrow X$ of class $\mathcal{U}$ such that $\mathrm{Dom}\Gamma \subset \mathrm{Dom}\overline{F}$.*

As before, the superscript "d" stands for "domain." Clearly, if Γ is $\mathcal{U}^d$-majorized, then it has an equilibrium $\widehat{x} \in X$; that is, $P_i(\widehat{x}) = \varnothing$ for each $i \in I$.

Lemma 5 *If a qualitative game $\Gamma = (X_i, P_i)_{i \in I}$ is $\mathcal{U}^d$-majorized, then it has an equilibrium.*

However, it is important to keep in mind, while constructing an appropriate F, that it might be quite difficult to satisfy the condition that $x \notin \overline{F}(x)$ for every $x \in X$.

The next result is an extension of Lemma 3 to qualitative games.

Corollary 2 *Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game and suppose that, for every $x \in \text{Dom}\Gamma$, there exist an open neighborhood of x in X, $U(x)$, and a collection of strict correspondences $D_i^x : U(x) \twoheadrightarrow X_i$, $i \in I$, such that*
(i) D_i^x is closed with convex values for each $i \in I$;
(ii) for each $A \in \langle \text{Dom}\Gamma \rangle$ and every $z \in \cap_{x \in A} U(x)$, there exists $i \in I$ such that $z_i \notin \text{co}\{\cup_{x \in A} D_i^x(z)\}$.
Then Γ has an equilibrium.

Proof. Assume, by contradiction, that Γ has no equilibrium. For each $x \in X$, consider $F^x : U(x) \twoheadrightarrow X$ defined by $F^x(z) = \Pi_{i \in I} D_i^x(z)$. By Lemma 3 of Fan (1952) (see also Aliprantis and Border 2006, Theorem 17.28), F^x is upper hemicontinuous on $U(x)$. Then Γ is $\mathcal{U}^d$-majorized by Lemma 3, and, therefore, it has an equilibrium by Lemma 5, a contradiction. ■

Remark It is useful to remember that a correspondence with compact Hausdorff range space is closed if and only if it is upper hemicontinuous and closed-valued (see, e.g., Aliprantis and Border 2006, Theorem 17.11). That is, in our framework, assuming that $D_i^x : U(x) \twoheadrightarrow X_i$ is closed is the same as assuming that D_i^x is an upper hemicontinuous correspondence with closed values.

From an applications' point of view, it might be beneficial to relax (i) of Corollary 2, since if for some i, $P_i(z) = \varnothing$ for all $z \in U_x$, then one would like to put $\text{Dom}D_i^x$ equal to the empty set in order to facilitate verifying (ii).

Theorem 2 *Let $\Gamma = (X_i, P_i)_{i\in I}$ be a qualitative game and suppose that, for each $x \in \text{Dom}\Gamma$, there exists $I(x) \subset I$, an open neighborhood of x in X, $U(x)$, and a collection of correspondences $D_i^x : U(x) \twoheadrightarrow X_i$, $i \in I$, such that*
(i) D_i^x is strict and closed with convex values for each $i \in I(x)$, and $\text{Dom}D_i^x = \varnothing$ for each $i \in I\backslash I(x)$;
(ii) for each $A \in \langle\text{Dom}\Gamma\rangle$ and every $z \in \cap_{x\in A}U(x)$, there exists $i \in \cup_{x\in A}I(x)$ such that $z_i \notin \text{co}\{\cup_{x\in A}D_i^x(z)\}$.
Then Γ has an equilibrium.

The proof of Theorem 2 is given in the Appendix.

4 Applications to Strategic Games

Consider a compact game $G = (X_i, u_i)_{i\in I}$ where $I = \{1, \ldots, n\}$ denotes the set of players, each player i's pure strategy set X_i is a nonempty, compact subset of a locally convex Hausdorff topological vector space, and each payoff function u_i is a bounded function from the Cartesian product $X = \Pi_{i\in N}X_i$, equipped with the product topology, to $\mathbb{R}$. Under these conditions, $G = (X_i, u_i)_{i\in I}$ is called a compact game. A game $G = (X_i, u_i)_{i\in I}$ is quasiconcave if each X_i is convex and $u_i(\cdot, x_{-i}) : X_i \to \mathbb{R}$ is quasiconcave for each $i \in N$ and every $x_{-i} \in X_{-i}$, where $X_{-i} = \Pi_{k\in N\backslash\{i\}}X_k$. Denote the set of all pure strategy Nash equilibria of G in X by E_G, the graph of G by $\text{Gr}G = \{(x, u) \in X \times \mathbb{R}^n \mid u_i(x) = u_i \text{ for all } i \in N\}$.

4.1 Correspondence Secure and Continuously Secure Games

In this section, we apply $\mathcal{U}^d$-majorized correspondences to studying equilibrium existence in correspondence secure games (Reny 2013) and continuously secure games (McLennan, Monteiro, and Tourky 2011; Barelli and Meneghel 2013).

The point secure games, due to Reny (2013), constitute a broad class of games that includes, in particular, the better-reply secure games (see, e.g., Prokopovych 2013, Section 3).

Definition 3 *A game* $G = (X_i, u_i)_{i \in I}$ *is point secure if whenever* $x \in X \backslash E_G$*, there exist an open neighborhood* $U(x)$ *of* x *in* X *and a strategy profile of deviation strategies* $d_i(x) = (d_1(x), \ldots, d_n(x)) \in X$ *such that, for every* $x' \in U(x) \backslash E_G$*, there exists a player* $i \in I$ *for whom* $u_i(d_i(x), z_{-i}) > u_i(x')$ *for every* $z \in U(x)$.

Correspondence security is a generalization of point security.

Definition 4 *A game* $G = (X_i, u_i)_{i \in I}$ *is correspondence secure if whenever* $x \in X \backslash E_G$*, there exist an open neighborhood* $U(x)$ *of* x *in* X*, a collection of closed correspondences* $D_i^x : U(x) \twoheadrightarrow X_i$*,* $i = 1, \ldots n$*, with nonempty and convex values such that, for every* $x' \in U(x) \backslash E_G$ *there exists a player* $i \in I(x)$ *for whom* $u_i(d_i, z_{-i}) > u_i(x')$ *for every* $z \in U(x)$ *and every* $d_i \in D_i(z)$.

Another, closely related generalization of better-reply security is continuous security (McLennan, Monteiro, and Tourky 2011; Barelli and Meneghel 2013).

Definition 5 *A game* $G = (X_i, u_i)_{i \in I}$ *is continuously secure if whenever* $x \in X \backslash E_G$*, there exist* $\alpha^x \in \mathbb{R}^n$*, an open neighborhood* $U(x)$ *of* x *and a collection of closed correspondences* $D_i^x : U(x) \twoheadrightarrow X_i$*,* $i = 1, \ldots, n$*, with nonempty and convex values such that:*

(a) $u_i(d_i, z_{-i}) \geq \alpha_i^x$ *for all* $i \in I$, $z \in U(x)$ *and* $d_i \in D_i^x(z)$*;*
(b) for every $x' \in U(x)$ *there exists* $i \in I$ *such that* $u_i(x') < \alpha_i^x$.

We now show, with the aid of Corollary 2, that every compact quasiconcave game that is correspondence secure has a pure strategy Nash equilibrium.

Theorem 3 *If* $G = (X_i, u_i)_{i \in I}$ *is compact, quasiconcave, and correspondence secure, then it possesses a pure strategy Nash equilibrium.*

Proof. Assume, by way of contradiction, that G has no Nash equilibrium in pure strategies. Since G is correspondence secure, for every $x \in X$ there exist an open neighborhood $U(x)$ of x in X, a collection of closed correspondences $D_i^x : U(x) \twoheadrightarrow X_i$, $i = 1, \ldots, n$, with nonempty and convex values such that, for every $x' \in U(x)$, there exists a player $i \in I$ for whom $u_i(d_i, z_{-i}) > u_i(x')$ for every $z \in U(x)$ and every $d_i \in D_i^x(z)$. Since X is compact, the open cover $\{U(x) : x \in X\}$ of X contains a finite subcover $\{U(x^j) : j \in J\}$, where J is a finite set. Let $\{C^j : j \in J\}$ be a closed refinement of $\{U(x^j) : j \in J\}$ such that $C^j \subset U(x^j)$ for each $j \in J$.

Consider some $x \in X$. Denote $J_C^x = \{j \in J : x \in C^j\}$. Define an open neighborhood $W(x)$ of x in X as follows:

$$W(x) = \cap_{j \in J_C^x} U(x^j) \cap (X \setminus \cup_{j \in J \setminus J_C^x} C^j).^7$$

For $W(x)$, we will define a collection of correspondences $D_i^{W(x)} : W(x) \twoheadrightarrow X_i$, $i \in I$, satisfying the hypotheses of Corollary 2. For each $i \in I$, it is possible to impose a weak ordering R_i^x on the set J_C^x: for any s, $t \in J_C^x$, $s R_i^x t$ (s is "weakly" preferred to t) if, for every $z \in U(x^s)$ and every $d_i^s(z) \in D_i^{x^s}(z)$, there exist $x' \in U_X(x^t)$ and $d_i^t(x') \in D_i^{x^t}(x')$ such that $u_i(d_i^s(z), z_{-i}) \geq u_i(d_i^t(x'), x'_{-i})$. One can check that this relation on J_C^x is complete, reflexive, and transitive. Since J_C^x is a finite set, the maximal

[7] For more details, see Reny (2013, Theorem 3.4)

set $M(R_i^x, J_C^x) = \{s \in J_C^x : sR_i^x t \text{ for each } t \in J_C^x\}$ is nonempty. For each $i \in I$, pick some $r_i \in M(R_i^x, J_C^x)$ and define $D_i^{W(x)} : W(x) \twoheadrightarrow X_i$ by $D_i^{W(x)}(z) = D_i^{x^{r_i}}(z)$ for every $z \in W(x)$. We need to show that for each finite subset A of X and every $z \in \cap_{x\in A} W(x)$, there exists $i \in I$ such that $z_i \notin \text{co}\{\cup_{x\in A} D_i^{W(x)}(z)\}$.

Assume, by contradiction, that there are a finite subset A of X and a point $\widetilde{z} \in \cap_{x\in A} W(x)$ such that $\widetilde{z}_s \in \text{co}\{\cup_{x\in A} D_s^{W(x)}(\widetilde{z})\}$ for each $s \in I$. Since $\widetilde{z} \in C^t$ for some $t \in J$, there exists $i \in I$ such that $u_i(d_i^t(z'), z'_{-i}) > u_i(\widetilde{z})$ for every $z' \in U(x^t)$ and every $d_i^t(z') \in D_i^{x^t}(z')$. By construction, for each $x \in A$, $D_i^{W(x)}$ possesses the following property: for every $z \in U(x^{r_i})$ and every $d_i^{W(x)}(z) \in D_i^{W(x)}(z)$, there exist $z' \in U(x^t)$ and $d_i^t(z') \in D_i^t(z')$ such that $u_i(d_i^{W(x)}(z), z_{-i}) \geq u_i(d_i^t(z'), z'_{-i})$. Then, for each $x \in A$, $u_i(d_i^{W(x)}(z), z_{-i}) > u_i(\widetilde{z})$ for every $z \in U(x^{r_i})$ and every $d_i^{W(x)}(z) \in D_i^{W(x)}(z)$. In particular, $u_i(d_i^{W(x)}(\widetilde{z}), \widetilde{z}_{-i}) > u_i(\widetilde{z})$ for each $x \in A$ and every $d_i^{W(x)}(\widetilde{z}) \in D_i^{W(x)}(z)$, which is impossible because $\widetilde{z}_i \in \text{co}\{\cup_{x\in A} D_i^{W(x)}(\widetilde{z})\}$ and u_i is quasiconcave in player i's own strategy. ■

It is not difficult to see that a continuously secure game is also correspondence secure.

Corollary 3 *If $G = (X_i, u_i)_{i\in I}$ is compact, quasiconcave, and continuously secure, then it possesses a pure strategy Nash equilibrium.*

Proof. Suppose, by way of contradiction, G has no pure strategy Nash equilibrium. Consider an arbitrary $x \in X$. Then there exist $\alpha^x \in \mathbb{R}^n$, an open neighborhood $U(x)$ of x and a collection of closed correspondences $D_i^x : U(x) \twoheadrightarrow X_i$, $i \in I$, with nonempty and convex values for which (a) and (b) of Definition 5 hold. Fix some $x' \in U(x)$. Then, there exists $i \in I$ such that $u_i(x') < \alpha_i^x \leq u_i(d_i, z_{-i})$ for all $z \in U(x)$ and $d_i \in D_i^x(z)$; that is, the game is also correspondence secure. ■

4.2 Correspondence Transfer Continuous Games

In this section, U^d-majorized correspondences are used for studying equilibrium existence in games that possess a property similar to generalized weakly transfer continuity.by Nessah (2011).

Definition 6 *A game $G = (X_i, u_i)_{i\in I}$ is correspondence transfer continuous if whenever $x \in X\backslash E_G$, there exist a player i, an open neighborhood $U_X(x)$ of x in X, and a closed correspondence $D_i^x : U_X(x) \twoheadrightarrow X_i$ with nonempty and convex values such that $u_i(d_i, x'_{-i}) > u_i(x')$ for every $x' \in U_X(x)\backslash E_G$ and every $d_i \in D_i^x(x')$.*

Theorem 4 follows from Theorem 2, a result subtler than Corollary 2.

Theorem 4 *If $G = (X_i, u_i)_{i\in I}$ is compact, quasiconcave, and correspondence transfer continuous, then it possesses a pure strategy Nash equilibrium.*

Proof. Assume, to the contrary, that there is a compact, quasiconcave, correspondence transfer continuous game, $G = (X_i, u_i)_{i\in I}$, with no pure strategy Nash equilibrium. Then, for every $x \in X$, there exist a player $i(x) \in I$, an open neighborhood $U(x)$ of x in X, and a closed correspondence $D_{i(x)}^x : U(x) \twoheadrightarrow X_{i(x)}$ with nonempty and convex values such that $u_{i(x)}(d_{i(x)}(x'), x'_{-i(x)}) > u_{i(x)}(x')$ for every $x' \in U(x)$ and every $d_{i(x)}(x') \in D_{i(x)}^x(x')$. Put $I(x) = \{i(x)\}$, and, for each $j \in I\backslash\{i(x)\}$, define $D_j^x : U(x) \twoheadrightarrow X_j$ as a correspondence with $\mathrm{Dom} D_j^x = \varnothing$. To be able to use Theorem 2, we need to show that for each $A \in \langle X\rangle$ and every $z \in \cap_{x\in A} U(x)$, there exists $i \in \cup_{x\in A} I(x)$ such that $z_i \notin \mathrm{co}\{\cup_{x\in A} D_i^x(z)\}$.

Assume that for some $A \in \langle X\rangle$ there exist $\widetilde{z} \in \cap_{x\in A} U(x)$ such that $\widetilde{z}_s \in \mathrm{co}\{\cup_{x\in A} D_s^x(\widetilde{z})\}$ for all $s \in \cup_{x\in A} I(x)$. Consider some $i \in \cup_{x\in A} I(x)$. Denote $A_i = \{x \in A : I(x) = \{i\}\}$. Then, for each $x \in A_i$, $u_i(d_i^x(x'), x'_{-i}) > u_i(x')$ for every $x' \in U(x)$ and every $d_i^x(x') \in D_i^x(x')$. In particular, it means that $u_i(d_i^x(\widetilde{z}), \widetilde{z}_{-i}) > u_i(\widetilde{z})$ for each $x \in A_i$ and every $d_i^x(\widetilde{z}) \in D_i^x(\widetilde{z})$, which is impossible because $\widetilde{z}_i \in \mathrm{co}\{\cup_{x\in A} D_i^x(\widetilde{z})\}$ and u_i is quasiconcave in player s's own strategy. ■

5 Conclusion

In this paper, we have used majorized correspondences to show equilibrium existence in discontinuous games. A new class of majorized correspondences, domain $\mathcal{U}$- majorized correspondences, is defined in a general, global fashion. Then, two ways of going from locally defined well-behaved correspondences to domain majorized correspondences are described.

The sufficient conditions for the existence of an equilibrium in qualitative games are formulated in a local fashion, for neighborhoods of points. The developed tools are used to show the existence of a pure strategy Nash equilibrium in compact quasiconcave games that are either correspondence secure or correspondence transfer continuous.

Appendix

Proof of Lemma 2

Suppose that F is locally $\mathcal{U}_\theta$-majorized. We have to show that it is $\mathcal{U}_\theta$-majorized. For every $x \in X$, there exists an open neighborhood $U(x)$ and a $\mathcal{U}_\theta$-majorant $F^x : U(x) \to Y$ of F at x such that $F(z) \subset F^x(z)$ and $\theta(z) \notin F^x(z)$ for every $z \in U(x)$. The open cover $\{U(x) : x \in X\}$ has a finite subcover $\{U(x_j) : j = 1, \ldots, J\}$. For each j, define $F'_j : X \to Y$ by

$$F'_j(z) = \begin{cases} F^{x_j}(z) \text{ if } z \in U(x_j), \\ Y \text{ if } z \in X \backslash U(x_j). \end{cases}$$

Since each F'_j is upper hemicontinuous on X and Y is a compact subset of a Hausdorff topological vector space, the correspondence $\overline{F} : X \twoheadrightarrow Y$ defined by $\overline{F}(z) = \cap_{j \in \{1,\ldots,J\}} F'_j(z)$ for every $z \in X$ is of class $\mathcal{U}_\theta$ (see, e.g., Aliprantis and Border 2006, Theorem 17.25). Clearly, $F(z) \subset \overline{F}(z)$ for every $z \in U(x)$.

Proof of Lemma 4

Since X is compact, the open cover $\{U(x) : x \in X\}$ of X contains a finite subcover $\{U(x_j) : j \in J\}$, where J is a finite set. For $z \in X$, denote $J_U^z = \{j \in J : z \in U(x_j)\}$ and $U^z = \cap_{j \in J_U^z} U(x_j)$ Define a correspondence $\overline{F} : X \twoheadrightarrow Y$ by $\overline{F}(z) = \cap_{j \in J_U^z} F^{x_j}(z)$ for every $z \in X$. Clearly, $\text{Dom}\overline{F} = X$.

We need to show that $\overline{F}$ is upper hemicontinuous. Fix some $z \in X$. Let W be a relatively open subset of Y such that $\overline{F}(z) \subset W$. Then, for each $j \in J_U^z$, there exists an open neighborhood U_j of z such that $U_j \subset U^z$ and $F^{x_j}(U_j) \subset W$. It is worth noticing that $\overline{F}(z') \subset \cap_{s \in J_U^z} F^{x_s}(z')$ for every $z' \in U^z$, which implies that $\overline{F}(z') \subset W$.

Proof of Theorem 2

Assume, by contradiction, that Γ has no equilibrium, i.e. $\text{Dom}\Gamma = X$. Since X is compact, the open cover $\{U(x) : x \in X\}$ of X contains a finite subcover $\{U(x_j) : j \in J\}$, where J is a finite set. Let $\{V_j : j \in J\}$ be an open refinement of $\{U(x_j) : j \in J\}$ such that $\text{cl}V_j \subset U(x_j)$ for every $j \in J$ (see, e.g., Urai 2010, Theorem 3.1.3). For each $j \in J$ and each $i \in I$, define a correspondence $\widehat{F}_i^j : X \twoheadrightarrow X_i$ by

$$\widehat{F}_i^j(z) = \begin{cases} D_i^{x_j}(z) \cup_{\{s \in J \setminus \{j\} : z \in \text{cl}V_s\}} D_i^{x_s}(z) \text{ if } z \in V_j \text{ and } i \in I(x_j), \\ X_i \text{ if } z \in X \setminus V_j \text{ or } i \in I \setminus I(x_j). \end{cases}$$

We need to show that each $\widehat{F}_i^j$ is upper hemicontinuous on X. Fix some $i \in I$, $j \in J$, and $z \in X$. Assume that there exists a relatively open, proper subset of X_i, W, such that $\widehat{F}_i^j(z) \subset W$. Clearly, $i \in I(x_j)$ and $z \in V_j$. Denote $J_C^z = \{s \in J : z \in \text{cl}V_s\}$. Since, for each $s \in J_C^z$, $D_i^{x_s}$ is either upper hemicontinuous or empty-valued at z, there exists a neighborhood U_s of z such that $U_s \subset V_j \cap U(x_s)$ and $D_i^{x_s}(U_s) \subset W$. Then $\widehat{F}_i^j(\cap_{s \in J_C^z} U_s \cap (\cap_{s \in J \setminus J_C^z}(X \setminus \text{cl}V_s))) \subset W$.

For $i \in I$ and $j \in J$, define $F_i^j : X \twoheadrightarrow Y$ by $F_i^j(z) = \text{co}\widehat{F}_i^j(z)$ for every

$z \in X$. Then F_j^i is compact-valued (see, e.g., Aliprantis and Border 2006, Lemma 5.29), and, moreover, F_i^j is upper hemicontinuous on X_i (see, e.g., Aliprantis and Border 2006, Theorem 17.35). Then, for each $i \in I$, the correspondence $F_i : X \twoheadrightarrow X_i$ defined by $F_i(z) = \cap_{j \in J} F_i^j(z)$ for every $z \in X$ is upper hemicontinuous on X. Therefore, the correspondence $\overline{F} : X \twoheadrightarrow X$ defined by $\overline{F}(z) = \Pi_{i \in I} F_i(z)$ is upper hemicontinuous on X.

Consider some $z \in X$. It lies in some V_j. Let us show that $z_{i'} \notin \mathrm{co} F_{i'}^j(z)$ for some $i' \in I(x_j)$. By (ii), there exists $i' \in \cup_{s \in J_C^z} I(x_s)$ such that $z_{i'} \notin \mathrm{co}\{\cup_{s \in J_C^z} D_{i'}^{x_s}(z)\}$. Since $\widehat{F}_{i'}^j(z) = \cup_{s \in J_C^z} D_{i'}^{x_s}(z)$, we have that $z_{i'} \notin \mathrm{co} F_{i'}^j(z)$. Therefore $z_{i'} \notin \mathrm{co} F_{i'}(z)$, and, consequently, $z \notin \mathrm{co}\overline{F}(z)$.

Since $\overline{F}$ is of class $\mathcal{U}$ and $\mathrm{Dom}\Gamma = \mathrm{Dom}\overline{F} = X$, Γ is $\mathcal{U}^d$-majorized. Then it has an equilibrium by Lemma 5, a contradiction.

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